Review Questions

Formally represent the following

- B has all the elements in A.
- A has no elements in B.
- Set of elements in both A and B.
- Some humans (H) are clubby (C).
- Clubby animals that are not humans.
- There are animals (A) that are neither human nor clubby animals.

B3: Functions

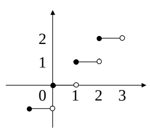
Today

- Formally represent functions and use them to analyze phenomena
  - Formally represent a function
  - Basic properties of function
  - Representing board games
- Take-home exercises
  - Corporate organization, Virtual pet, Board game

Need for practice

Unary Numeric Functions

- factorial $n!$: $n \times (n-1) \times (n-2) \times ... \times 2 \times 1$
- floor $\lfloor x \rfloor$: The greatest integer $\leq x$



- ceiling $\lceil x \rceil$: The smallest integer $\geq x$

n-ary Numeric Functions

- remainder m mod n: The remainder when m is divided by n (binary)
- Solutions of a quadratic equation $ax^2 + bx + c = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

(ternary)

Two values?

Non-Numeric Functions

- ASCII values
  - 'a' $\rightarrow$ 97
  - 'b' $\rightarrow$ 98

  etc.
- Morse code
  - 'a' $\rightarrow$ "· –"
  - 'b' $\rightarrow$ "– · · ·"

  etc.

Formal definition?
Connection to relations?

Formal Definition (Unary)

- *ascii* = {('**a**', 97), ('**b**', 98), ...}
- *factorial* = {(0, 1), (1, 1), (2, 2), (3, 6), ...}

  $= \{(x, y) \mid y = x \times (x-1) \times (x-2) \times ... \times 2 \times 1\}$
- *floor*

  = {(0, 0), (0.1, 0), ..., (0.9999, 0), (1, 1), ...}

  = {(x, y) | y is the greatest integer such that $y \leq x$}

Binary (n-ary) functions?

Formal Definition (Binary)

- *remainder* — cf. ternary relation
  $= \{((0, 2), 0), ((1, 2), 1), ((2, 2), 0), ...\}$
  $= \{((x, y), z) \mid z$ is the remainder of dividing x by $y\}$
- Notes
  - Input and output are represented as elements of the input-output pair.
  - If the input consists of multiple objects, it is represented as a pair/n-tuple (embedded in the input-output pair).

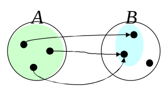
Function vs. Relation

- Very similar representation, but
  - Functions have clear *separation between input and output*.
  - Functions often represent *operations*.
  - Use of function $\Rightarrow$ Results in an object
    - E.g., set operation returns a set
  - Use of relation $\Rightarrow$ Results in true/false
    - E.g., set relation means true/false

Function Type

- Unary function f from set A to set B
  - $f: A \to B$

  cf. $R: A \times B$



- Binary function g from Cartesian sets A and B to set C
  - $g: A \times B \to C$

  Schematic?
- n-ary function (analogous)

Using Functions

- Unary function $f: A \to B$
  - $f = \{(a, b), ...\}$
  - $(a, b) \in f$
  - $f(a) = b$ [commonly used in math]
- Binary function $g: A \times B \to C$
  - $g = \{((a, b), c), ...\}$
  - $((a, b), c) \in g$
  - $g(a, b) = b$

Function, Formally

- Unary function: Special kind of binary relation $f \subseteq A \times B$ where:
  - For *every* $a \in A$, there is *exactly one* b such that $f(a) = b$. Type: $f: A \to B$
- Binary function: Special kind of ternary relation $g \subseteq (A \times B) \times C$ where:
  - For *every* $(a, b) \in A \times B$, there is *exactly one* c such that $g(a, b) = c$. Type: $g: A \times B \to C$
- n-ary function: Special kind of $(n + 1)$-ary relation

0-ary functions?

Section Summary

- Function is a special case of relation.
- Function must map *all the elements* in the input set.
- Function must map every element *to exactly one element* in the output set.
- Function type: e.g., $f: A \to B$
- Function definition: e.g., $f = \{(a, b), ...\}$

Group Exercise

1. Define a function *succ* that adds 1 to the given natural number (**N** = the set of natural numbers).
2. Explain why *succ* is a function.
3. Formally represent that 2 is the result of applying *succ* twice to 0.
4. Can you define a function *prev* that subtracts 1 from the given natural number? Explain.

Inverse Functions

- Movement of a board game

Reverse operation?

- $f(x) = 2x + 1$
- $g(x) = 2^x$
- $h(x) = x(x + 1)(x - 1)$
- $i(x) = \sin x$

Graphical characterization of inverse?

Properties of Function

Function f: $A \rightarrow B$

- Surjective/Onto: For *every* $b \in B$, there is *at least one* $a \in A$ such that $(a, b) \in f$
- Injective/One-to-one: For *every* $b \in B$, there is *at most one* $a \in A$ such that $(a, b) \in f$
- Bijective: Both surjective and injective

Schematic?

Group Exercise

- Are the following functions surjective, injective, or bijective?

  A. $f(x) = \sin x$

  B. $g(x) = x \sin x$

  C. $h(n) = 2n$ [$n \in$ the set of natural numbers]

  D. $i(x) = 2x$ [$x \in$ the set of real numbers]

Inverse of a Relation

- Inverse of $R \subseteq A \times B$: $\{(b, a) \mid (a, b) \in R\}$
  - Notation: R^{-1}
- Example
  - $A = \{a, b\}$, $B = \{1, 2, 3\}$
  - $R = \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)\}$
  - $R^{-1} = \{(1, a), (2, a), (3, a), (1, b), (2, b), (3, b)\}$

cf. complement, R'

Inverse of a Function

- Definition/notation same as for relations
- Warning: The inverse of a function may or may not be a function (still a relation).
  - Must cover all the input
  - Must have a unique output
- Examples
  - $f(x) = x + 2$, $f^{-1}(x) = x - 2$
  - $f(x) = x^2$, $f^{-1}(x)$ does *not* exist

Section Summary

- The properties, *surjective*, *injective*, and *bijective*, can be used for classifying functions and also for identifying the applicability of inverse.
- Only bijective functions have their *inverses*.

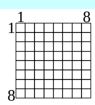
Group Exercise

- To represent the position of a hurricane at a specific time as a function h.
  - Time: Use the number of seconds after the hurricane is born.
  - Position: Use latitude and longitude (as real numbers, **R**).
- Identify the "type" of h.
- Define h for some hypothetical scenario.
- Classify h: surjective, injective, bijective?
- Does h have an inverse?

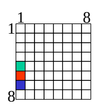
Section 3

Board Games

- Example: Chess board
  - $Side = \{1, 2, ..., 8\}$
  - $Board = ?$



- Function $move_{pawn} = ?$ Type? Formal definition?
- Function use examples
  - $move_{pawn}$?



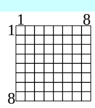
Composition

- Given $f: A \rightarrow B$ and $g: B \rightarrow C$, the function from A to C $\{(a, c) \mid$ there is $b \in B$ such that $f(a) = b$ and $g(b) = c\}$
  - Notation: $g \circ f$ Schematic?
- Examples
  - $(succ \circ succ)(0) = succ^2(0) = succ(succ(0))$
  - $f(x) = 2x$, $g(x) = x + 1$, $(g \circ f)(x) = g(f(x))$ $= 2x + 1$

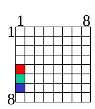
Composition & Tuple-valued function

Board Games

- Example: Chess board
  - $Side = \{1, 2, ..., 8\}$
  - $Board = Side \times Side$



- Function $move_{pawn} = \{((7, 1), (6, 1)), ((7, 2), (6, 2)), ...\}$ (input: $(7, 1)$, output: $(6, 1)$)
- Examples
  - $move_{pawn}(7, 1) = (6, 1)$
  - $move_{pawn}(move_{pawn}(7, 1)) = (5, 1)$



Section Summary

- Function outputs can be an n-tuple.
- Where the input/output interface is appropriate, functions can be composed to obtain a complex function.

Summary Exercise

- For what would you use function composition?
  - Hint: Informally, what does it correspond to?
- Questions/Comments/Suggestions