

## Review Questions

Formally represent the following

- B has all the elements in A.
- A has no elements in B.
- Set of elements in both A and B.
- Some humans (H) are clubby (C).
- Clubby animals that are not humans.
- There are animals (A) that are neither human nor clubby animals.

## B3: Functions

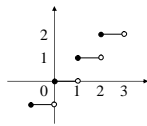
*Today*

- Formally represent functions and use them to analyze phenomena
  - Formally represent a function
  - Basic properties of function
  - Representing board games
- Take-home exercises
  - Corporate organization, Virtual pet, Board game

### Section 1

## Unary Numeric Functions

- **factorial**  $n!$ :  $n \times (n - 1) \times (n - 2) \times \dots \times 2 \times 1$
- **floor**  $\lfloor x \rfloor$ : The greatest integer  $\leq x$



- **ceiling**  $\lceil x \rceil$ : The smallest integer  $\geq x$

## $n$ -ary Numeric Functions

- **remainder**  $m \bmod n$ : The remainder when  $m$  is divided by  $n$  (binary)

- Solutions of a quadratic equation

$$ax^2 + bx + c = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

(ternary)

Two values?

## Non-Numeric Functions

- ASCII values
  - 'a'  $\rightarrow$  97
  - 'b'  $\rightarrow$  98
  - etc.
- Morse code
  - 'a'  $\rightarrow$  ". -"
  - 'b'  $\rightarrow$  "- . . ."
  - etc.

Formal definition?  
Connection to relations?

## Formal Definition (Unary)

- $ascii = \{('a', 97), ('b', 98), \dots\}$
- $factorial = \{(0, 1), (1, 1), (2, 2), (3, 6), \dots\}$   
 $= \{(x, y) \mid y = x \times (x - 1) \times (x - 2) \times \dots \times 2 \times 1\}$
- $floor$   
 $= \{(0, 0), (0.1, 0), \dots, (0.9999, 0), (1, 1), \dots\}$   
 $= \{(x, y) \mid y \text{ is the greatest integer such that } y \leq x\}$

Binary ( $n$ -ary) functions?

## Formal Definition (Binary)

- *remainder* cf. ternary relation  
=  $\{((0, 2), 0), ((1, 2), 1), ((2, 2), 0), \dots\}$   
=  $\{(x, y), z \mid z \text{ is the remainder of dividing } x \text{ by } y\}$
- Notes
  - Input and output are represented as elements of the input-output pair.
  - If the input consists of multiple objects, it is represented as a pair/ $n$ -tuple (embedded in the input-output pair).

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## Function vs. Relation

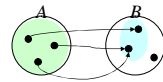
- Very similar representation, but
  - Functions have clear *separation between input and output*.
  - Functions often represent *operations*.
  - Use of function  $\Rightarrow$  Results in an **object**
    - E.g., set operation returns a set
  - Use of relation  $\Rightarrow$  Results in **true/false**
    - E.g., set relation means true/false

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## Function Type

- **Unary** function  $f$  from set  $A$  to set  $B$ 
  - $f: A \rightarrow B$
  - cf.  $R: A \times B$
- **Binary** function  $g$  from Cartesian sets  $A$  and  $B$  to set  $C$ 
  - $g: A \times B \rightarrow C$  Schematic?
- $n$ -ary function (analogous)



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## Using Functions

- **Unary** function  $f: A \rightarrow B$ 
  - $f = \{(a, b), \dots\}$
  - $(a, b) \in f$
  - $f(a) = b$  [commonly used in math]
- **Binary** function  $g: A \times B \rightarrow C$ 
  - $g = \{((a, b), c), \dots\}$
  - $((a, b), c) \in g$
  - $g(a, b) = c$

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## Function, Formally

- **Unary function:** Special kind of binary relation  $f \subseteq A \times B$  where:
  - For **every**  $a \in A$ , there is **exactly one**  $b$  such that  $f(a) = b$ . Type:  $f: A \rightarrow B$
- **Binary function:** Special kind of ternary relation  $g \subseteq (A \times B) \times C$  where:
  - For **every**  $(a, b) \in A \times B$ , there is **exactly one**  $c$  such that  $g(a, b) = c$ . Type:  $g: A \times B \rightarrow C$
- $n$ -ary function: Special kind of  $(n + 1)$ -ary relation

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0-ary functions?

## Section Summary

- Function is a special case of relation.
- Function must map **all the elements** in the input set.
- Function must map every element **to exactly one element** in the output set.
- Function type: e.g.,  $f: A \rightarrow B$
- Function definition: e.g.,  $f = \{(a, b), \dots\}$

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## Group Exercise

1. Define a function *succ* that adds 1 to the given natural number ( $\mathbf{N}$  = the set of natural numbers).
2. Explain why *succ* is a function.
3. Formally represent that 2 is the result of applying *succ* twice to 0.
4. Can you define a function *prev* that subtracts 1 from the given natural number? Explain.

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Section 2

## Inverse Functions

- Movement of a board game

Reverse operation?

- $f(x) = 2x + 1$
- $g(x) = 2^x$
- $h(x) = x(x + 1)(x - 1)$
- $i(x) = \sin x$

Graphical characterization of inverse?

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## Properties of Function

Function  $f: A \rightarrow B$

- **Surjective/Onto:** For *every*  $b \in B$ , there is *at least one*  $a \in A$  such that  $(a, b) \in f$
- **Injective/One-to-one:** For *every*  $b \in B$ , there is *at most one*  $a \in A$  such that  $(a, b) \in f$
- **Bijective:** Both surjective and injective

Schematic?

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## Group Exercise

- Are the following functions surjective, injective, or bijective?

A.  $f(x) = \sin x$

B.  $g(x) = x \sin x$

C.  $h(n) = 2n$  [ $n \in$  the set of natural numbers]

D.  $i(x) = 2x$  [ $x \in$  the set of real numbers]

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## Inverse of a Relation

- Inverse of  $R \subseteq A \times B$ :  $\{(b, a) \mid (a, b) \in R\}$ 
  - Notation:  $R^{-1}$
- Example
  - $A = \{a, b\}$ ,  $B = \{1, 2, 3\}$
  - $R = \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)\}$
  - $R^{-1} = \{(1, a), (2, a), (3, a), (1, b), (2, b), (3, b)\}$  **cf. complement,  $R'$**

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## Inverse of a Function

- Definition/notation same as for relations
- **Warning:** The inverse of a function may or may not be a function (still a relation).
  - Must cover all the input
  - Must have a unique output
- Examples
  - $f(x) = x + 2$ ,  $f^{-1}(x) = x - 2$
  - $f(x) = x^2$ ,  $f^{-1}(x)$  does *not* exist

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## Section Summary

- The properties, *surjective*, *injective*, and *bijective*, can be used for classifying functions and also for identifying the applicability of inverse.
- Only bijective functions have their *inverses*.

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## Group Exercise

- To represent the position of a hurricane at a specific time as a function  $h$ .
  - Time: Use the number of seconds after the hurricane is born.
  - Position: Use latitude and longitude (as real numbers,  $\mathbf{R}$ ).
- Identify the “type” of  $h$ .
- Define  $h$  for some hypothetical scenario.
- Classify  $h$ : surjective, injective, bijective?
- Does  $h$  have an inverse?

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### Section 3

## Board Games

- Example: Chess board
  - Side =  $\{1, 2, \dots, 8\}$
  - Board = ?
- Function  $move_{pawn} = ?$  Type? Formal definition?
- Function use examples
  - $move_{pawn}$  ?



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## Composition

- Given  $f: A \rightarrow B$  and  $g: B \rightarrow C$ , the function from  $A$  to  $C$   $\{(a, c) \mid \text{there is } b \in B \text{ such that } f(a) = b \text{ and } g(b) = c\}$ 
  - Notation:  $g \circ f$  Schematic?
- Examples
  - $(succ \circ succ)(0) = succ^2(0) = succ(succ(0))$
  - $f(x) = 2x, g(x) = x + 1, (g \circ f)(x) = g(f(x)) = 2x + 1$

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### Composition & Tuple-valued function

## Board Games

- Example: Chess board
  - Side =  $\{1, 2, \dots, 8\}$
  - Board =  $Side \times Side$
- Function  $move_{pawn} = \{((7, 1), (6, 1)), ((7, 2), (6, 2)), \dots\}$  input    output
- Examples
  - $move_{pawn}(7, 1) = (6, 1)$
  - $move_{pawn}(move_{pawn}(7, 1)) = (5, 1)$



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## Section Summary

- Function outputs can be an  $n$ -tuple.
- Where the input/output interface is appropriate, functions can be *composed* to obtain a complex function.

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## Summary Exercise

- For what would you use function **composition**?
  - Hint: Informally, what does it correspond to?
- [Questions/Comments/Suggestions](#)