## Review Questions

1. How to justify contrapositive?
   i.e., j ® y ⇔ Øy ® Øj
2. Prove $r \rightarrow \neg c$ from the hypothesis $\neg r$?
3. Prove $\neg r$ from the hypotheses $r \rightarrow c$ and $\neg c$?



B5 Use as reference

## Analysis Techniques (1)

- Identify special conditions on subformulas
  - j Ù y: To be **T**, both j and y must be **T**
  - j Ú y: To be **F**, both j and y must be **F**
  - j ® y: To be **F**, j must be **T** and y must be **F**
  - j « y: To be **T**, j and y must agree
- Using a new formula without affecting value
  - If j is **T**, j Ú y and y Ú j must be **T** for *any* y



B5 Use as reference

## Analysis Techniques (2)

- Use of equivalence
  - ØØj is equivalent to j — double negation
  - j Ú y ⇔ y Ú j — commutativity
  - j Ù y ⇔ y Ù j
  - Ø(j Ú y) ⇔ Øj Ù Øy — 'De Morgan'
  - Ø(j Ù y) ⇔ Øj Ú Øy
  - Øj Ú y ⇔ j ® y — '→' shorthand
  - j « y ⇔ (j ® y) Ù (y ® j) — '↔' shorthand
  - j ® y ⇔ Øy ® Øj — 'contrapositive'



## B7: Logic and Structures

*Today*

- Understand how to use FOL to specify structure
  - Review FOL symbols/syntax
  - FOL semantics and logic-structure connection
- B7 Supplemental Exercises with solutions available on-line



Section 1

## Objects in a Room [A1]

- An object must have another object on top of it.
- An object cannot be on top of itself.
- If an object $X$ is on top of another object $Y$, $Y$ cannot be on top of $X$.

***onTop*(_, _)**



## Group Exercise 1

- Natural number arithmetic
  - Constant symbol: **0**
  - Unary function symbol: ***succ***
  - Binary function symbol: +
  - Binary relation symbol: £
- Express the following in FOL:
  - a. "There is an element ≤ to any element"
  - b. "There are at least two elements ≤ 1" — represent 1?
  - c. "Between any two consecutive numbers, there exist no numbers"

Section 2

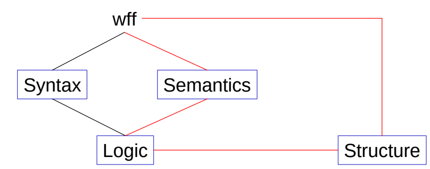
## Logic-Structure Connection

- Explain?
  - Predicate/relation symbol (FOL) vs. relation (structure)
  - Function symbol (FOL) vs. function (structure)
  - Symbol in FOL corresponding to set in structure

schematic?



## Overview: FOL Semantics




- Structures satisfy a wff (or collection of wff's).
- Wff's specify structure.



## Semantic Components

- Satisfaction: Relation between structures and wff's
- Interpretation: The semantics of the logic proper, i.e., logical connectives and other symbols
- Interpretation of user-defined symbols

| *Logic* | *Structure* |
|---|---|
| – Constant symbol *c* | An element in the set |
| – Binary function symbol *f* | A binary function *f* |
| – Binary predicate symbol *p* | A binary relation *p* |
| (symbols) | (more direct representation) |



symbol (syntax) vs. meaning (semantics)

## Satisfaction

- A structure $S$ and a wff $f$
  - E.g., $S$ = (*Individuals*, *Kicks*), $f$: $k(j, m)$

  $S$ satisfies $f$

  ⇔ $f$ is true in $S$

  ⇔ $f$ is satisfiable in $S$

Need a structure to analyze true/false
cf. propositional logic

- A structure $S$ and a set of wff's $F$
  - E.g., $S$ = (*Individuals*, *Kicks*), $F = \{k(j, m), k(m, j)\}$

  $S$ satisfies $F$

  ⇔ $F$ specifies $S$



## Example: Kicking Family

- **KF** = (*Individuals*, *Kicks*, *SpouseOf*)
  - *Individuals* = {John, Mary}
  - *Kicks* = {(John, Mary)}
  - *SpouseOf* = {(John, Mary), (Mary, John)}

schematic?

- Satisfiable in the above structure?
  - ***k*(*j*, *m*)** Ù ***k*(*m*, *j*)**
  - $***x*** **(*x* = *s*(*m*)** Ù ***k*(*x*, *m*))**
  - "***x*** **(*k*(*j*, *x*)** ® ***x* = *m*)**
  - Ø$***x*** "***y k*(*x*, *y*)**

***j*** ("John"), ***m*** ("Mary")
***k*** ("*Kicks*")
***s*** ("*SpouseOf*")

equivalent statements?



## Semantic Analysis (1)

- Connectives: ¬, ∧, ∨, →, ↔
  - Semantic table (propositional logic)
- Individual constants: e.g., ***j*** ("John"), ***m*** ("Mary")
  - Interpretation of the symbol, i.e., some individual
- Predicate/relation symbols ($n$-ary): applies to $n$ individuals, e.g., ***k*(*m*, *j*)**
  - Whether the relation (in the structure) holds for the individuals
  - E.g., ***k*(*m*, *j*)** is satisfied if (Mary, John) ∈ *Kicks*



relation symbol (syntax) vs. relation (semantics)

## Semantic Analysis (2)

- Function symbols (*n*-ary): e.g., ***s***
  - The return value of the function (in the structure), i.e., some individual
  - E.g., ***s*(*m*)** is John if (Mary, John) ∈ *SpouseOf*
- Equality symbol: =
  - Whether the individuals on both sides are identical
  - E.g., ***m*** = ***s*(*j*)** is satisfied if Mary = *SpouseOf*(John)

function symbol (syntax) vs. function (semantics)



## Semantic Analysis (3)

- Universal quantifier and individual variables: e.g., "***x***
  - "***x*** j : Whether substitution of every individual to ***x*** would satisfy j
  - E.g., "***x k*(*m*, *x*)** is satisfied if (Mary, John) ∈ *Kicks* and (Mary, Mary) ∈ *Kicks* (i.e., everyone in *Individuals*)
- Existential quantifier and individual variables: e.g., $***x***
  - $***x*** j : Whether subsitution of at least one individual to ***x*** would satisfy j
  - E.g., $***x*** (***x*** = ***s*(*m*)**) is satisfied if John = *SpouseOf*(Mary) (i.e., someone in *Individuals*)



## Semantic Analysis (4)

- Complex statements ⇒ Analyze top-down
- Example: "***x*** $***y*** (***x*** = ***y***) in **KF** — cf. bottom-up
  - Let j be $***y*** (***x*** = ***y***), and analyze "***x*** j
    - Analyze j with every individual for ***x*** : $***y*** (John = ***y***), $***y*** (Mary = ***y***)
    - For each case above (e.g., for John = ***y***),
      - Let j be John = ***y***, and analyze $***y*** j
      - Find some individual for ***y***: John = John
      - This case is satisfied in **KF**. — other case?



## Group Exercise 2

- **KF2** = (*Individuals*, *Kicks*, *SpouseOf*)
  - *Individuals* = {John, Mary}
  - *Kicks* = {(Mary, Mary), (Mary, John)}
  - *SpouseOf* = {(John, Mary), (Mary, John)}
- Satisfiable in the above structure?
  - $***x*** "***y k*(*x*, *y*)**
  - "***x*** (***x*** = ***s*(*m*)** ® $***y k*(*x*, *y*)**)
  - "***x*** "***y*** "***z*** ((***k*(*x*, *y*)** Ù ***k*(*y*, *z*)**) ® ***k*(*x*, *z*)**)

***j*** ("John"), ***m*** ("Mary")
***k*** ("*Kicks*")
***s*** ("*SpouseOf*")

Should we check all the possibilities?



## Constants

- Constant symbol = 0-ary function symbol
- Interpretation: the fixed return value of the 0-ary function
- Example
  - Symbols: ***j***, ***m***
  - **KF** = (*Individuals*, *J*, *M*, *Kicks*, *SpouseOf*)
    - *Individuals* = {John, Mary}
    - *J*: → *Individuals*, *M*: → *Individuals*
    - *J* = John, *M* = Mary
  - Interpretation: ***j*** interpreted as *J*, ***m*** as *M*

Often not shown explicitly (assuming a certain association)



## Definition of Limit [Calculus]

$$\lim_{x \to a} f(x) = c$$

- For every real number $\varepsilon > 0$, there exists a real number $\delta > 0$ such that $|f(x) - c| < \varepsilon$ whenever $0 < |x - a| < \delta$.

schematically?

## Group Exercise 3

- In the context of Kicking Family, find the *smallest* structure that would satisfy the following statements:

1. **($x k(x, m)) Ù ($x k(x, j))**
2. **Ø(j = m)**
3. **"x $y (y = s(x))**
4. **"x "y "z ((y = s(x) Ù z = s(x)) ® y = z)**
5. **"x "y (y = s(x) ® Øk(x, y))**

***j*** (“John”), ***m*** (“Mary”)
***k*** (“*Kicks*”)
***s*** (“*SpouseOf*”)

Hint: Start from a schematic



## Group Exercise 4

- In the context of Kicking Family, find the *smallest* structure that would satisfy the following statements:

1. **$x (k(j, x) Ù k(m, x))**
2. **"x $y (y = s(x) Ù k(y, x))**
3. **"x "y ((k(x, y) Ù k(y, x)) ® x = y)**

***j*** (“John”), ***m*** (“Mary”)
***k*** (“*Kicks*”)
***s*** (“*SpouseOf*”)

name?

Analogous scheme in a movie?



## Group Exercise 5

- Find a structure (with necessary components) that would satisfy the following statements:

1. **"x "y (x < y ® Ø(y < x))**
2. **"x "y "z ((x < y Ù y < z) ® x < z)**
3. **"x Ø(x < 0)**
4. **"x "y ((x < y Ú x = y) « x < f(y))**
5. **"x (x ¹ 0 ® $y (x = f(y)))**

names?

Constant symbol: **0**
Unary function symbol: ***f***
Binary relation symbol: <



## Section Summary

- Semantics: Ability to identify the truth value of a FOL statement with respect to a certain structure
  - A statement is true (in the given structure) if the structure satisfies the statement.
  - Otherwise, the statement is false.
- A collection of statements can be used to specify structures (that would satisfy all those statements).



## Module B Summary

- Logic-structure connection, formally represented
  - We can transform a variety of real-world problems into computational problems (formal representation).
  - We gain preciseness and conciseness.
  - Easier to tackle the problem mathematically or computationally.

Structure patterns commonly used in Computer Science?



## Module B Evaluation Workshop

- Fri., Oct. 17, class time (next class)
- Review the relevant part of the syllabus and the on-line handbook
- Re-use your manila folder or large envelope (or prepare one)
- Complete and bring “Take-Home Exercise Self-Evaluation Form” (distributed today) along with exercises
- Exercise B6 will be returned that day. Include it in the pile then.
- Complete and bring “Module B Comprehensive Exercises” (available on-line)
- Group evaluation sessions (open book): 20 min × 3
- “Comprehensive Exercise Self-Evaluation Forms” will be distributed that day (no need to print in advance)
- Submit all materials ***including mini project*** at the end of the session

## Summary Exercise

- Find the *smallest* structure that would satisfy the following statement:
  - $***x* (*x* = *x*)**
- Questions/Comments/Suggestions