Unit B7: Supplemental Exercises, 10/13/03

Notes:

- You do ***not*** need to submit these exercises. However, they may be helpful when you do Module B Comprehensive Exercises. Sample solutions are also provided. Use them wisely. Being able to understand provided solutions rarely guarantees that you can solve similar problems on your own.
- These exercises must appear out of season because they are from Spring 2003.

Exercise 1: Spring Break

The bad news is that your spring break is over. The good news is that you can turn your spring break into a more memorable *form* (i.e., formal representation). First, consider the following symbols:

- Unary predicate/relation symbols: *leisure*, *academic*
- Binary predicate/relation symbol: *moreFun*

Let us suppose that the following logical statements would specify your spring break:

Conditions:

1. $\exists x \exists y\ (leisure(x) \wedge leisure(y) \wedge x \neq y)$
2. $\exists x\ academic(x)$
3. $\forall x\ (leisure(x) \vee academic(x))$
4. $\forall x \forall y\ (((leisure(x) \wedge academic(y)) \rightarrow moreFun(x, y))$
5. $\forall x \forall y\ ((leisure(x) \wedge leisure(y)) \rightarrow \neg(moreFun(x, y) \vee moreFun(y, x)))$

Consider a structure **SpringBreak** = (*Activities*, *Leisure*, *Academic*, *MoreFun*) where

- *Activities* contains all sorts of activities, e.g., leisure and academic activities.
- *Leisure* and *Academic* are subsets of *Activities*, and interpret the unary predicate symbols *leisure* and *academic*, respectively. For example, *leisure*(*a*) is true if an activity *a* is a member of the set *Leisure*.
- *MoreFun* defines the meaning of the predicate symbol *moreFun*. E.g., *moreFun*(*a*, *b*) is true if $(a, b) \in moreFun$, i.e., you have more fun with *a* than with *b*.

A. Translate Condition 5 to a statement in plain English. Do not give a literal translation of logical symbols. Make your statement as easy as possible even to folks not in this class (of course, with exactly the same meaning).

B. Could there be activities other than leisure or academic? **Explain**.

C. Condition 3 alone does not exclude (potentially counterintuitive) possibilities of (1) $Academic \subseteq Leisure$ and (2) $Leisure \subseteq Academic$, because logical connective '$\vee$' is not exclusive. Fully analyze both Case 1 and Case 2 with respect to their possibilities, by referring to relevant logical statement(s).

D. Formally define all the structure components of the *smallest* instance of **SpringBreak** that would satisfy all the conditions listed above. For relations/functions, **give their types** as well.

Note: The smallest structure must contain the least possible number of elements in set(s), relation(s), and function(s), where applicable.

Answers:

A. All leisure activities are equally fun.

B. No. Condition 3 specifies that all activities are either leisure or academic.

C. Case 1 $Academic \subseteq Leisure$

Then, $a \in Academic$ is also in *Leisure*. Then, by Condition 4, a (as a member of *Leisure*) is moreFun than a (as a member of *Academic*). Although it sounds ridiculous, it does not violate most conditions just by itself. But Condition 5 says that all leisure activities are equally fun, prohibiting the situation a is moreFun than a as well. thus, this case cannot happen.

Case 2 $Leisure \subseteq Academic$

Then, $l \in Leisure$ is also in *Academic*. If there are two leisure activities, say l_1 and l_2, both of them are also academic activities. But then, these activities can satisfy $leisure(l_1)$ and academic(l_2), leading to the situation where l_1 is more fun than l_2 by Condition 4. This again violates Condition 5.

D. $Activities = Leisure \cup Academic$

$Leisure = \{l_1, l_2\}$

$Academic = \{a\}$

$MoreFun: Activities \times Activities$

$MoreFun = \{(l_1, a), (l_2, a)\}$

Exercise 2: North Pole (out of season)

In this problem, we will consider a hypothetical, out-of-season or off-duty north pole establishment that can be specified by logical statements involving the following symbols:

- Unary predicate/relation symbols: *santa*, *reindeer*
- Binary predicate/relation symbol: *feed*

Here are the logical statements:

Conditions:

1. $\neg\exists x\, (\neg reindeer(x) \wedge \neg santa(x))$
2. $\exists x\, santa(x)$
3. $\forall x \exists y\, feed(y, x)$
4. $\forall x\, \neg\, feed(x, x)$
5. $\forall x\, (santa(x) \rightarrow \neg\exists y\, feed(x, y))$
6. $\forall x \forall y\, (feed(x, y) \rightarrow \neg\, feed(y, x))$

Consider a structure **NorthPole** = (*Universe*, *Santa*, *Reindeer*, *Feed*) where

- *Universe* contains all sorts of elements, e.g., possibly Santa Claus and reindeer.
- *Santa* and *Reindeer* are subsets of *Universe*, and interpret the unary predicate symbols *santa* and *reindeer*, respectively. For example, *reindeer*(a) is true if an object a is a member of set *Reindeer*.
- *Feed* defines the meaning of the predicate symbol *feed*, possibly applicable to both Santa Claus and reindeer. For example, *feed*(a, b) is true if (a, b) $\in$ *feed*, i.e., a feeds b.

A. Transform Condition 1 so that there would be no negation ($\neg$) involved in the statement. Then, explain Condition 1 based on the transformed form.

   Hint: Review Section 2 of Lecture B6 slides.

B. Explain whether there must be at least one reindeer. Refer to relevant condition(s).

C. Explain whether there must be at least two reindeer. Refer to relevant condition(s).

D. Explain whether there must be at least three reindeer. Refer to relevant condition(s).

E. Formally define all the structure components of the *smallest* instance of **NorthPole** that would satisfy all the conditions listed above. For relations/functions, **give their types** as well.

Answers:

A. $\forall x\ (santa(x) \vee reindeer(x))$. Any individual in the universe is either Santa Claus or a reindeer.

B. Yes. There must be at least one Santa Claus (Condition 2). Everyone including the Santa Claus must be fed (Condition 3). But nobody including the Santa Claus can feed themselves (Condition 4). Furthermore, Santa Claus cannot feed anyone (Condition 5). Thus, there must be a reindeer, which can feed the Santa Clause.

C. Yes. Again, everyone including the reindeer must be fed (Condition 3). Since the reindeer cannot feed itself (Condition 4) and Santa Claus cannot feed reindeer (Condition 5), there must be another reindeer.

D. Yes. The two reindeer cannot feed each other (Condition 6). Again, Santa Claus cannot feed reindeer (Condition 5). Thus, there must be the third reindeer.

E. $Universe = Santa \cup Reindeer$

   $Santa = \{s\}$

   $Reindeer = \{\ d_1, d_2, d_3\ \}$

   $Feed: Universe \times Universe$

   $Feed = \{(d_1, s), (d_2, d_1), (d_3, d_2), (d_1, d_3)\}$

<End>