Unit C1: Operational and Relational Structures, 10/23/03

Exercise 1: Module B Comprehensive Exercises Review

Recall the general comments on comprehensive exercises made at the beginning of the class, and concisely reflect on them.

If you have questions and/or uncertainty, list them. Note that the list **must be your own** because everyone must have different ideas. This should help if you review your exercises individually with the instructor for possible upgrading.

Exercise 2: Mini Project Phase 1 Review

Recall the general comments on mini projects made at the beginning of the class, and concisely reflect on them.

Exercise 3: Set Operations

Note: In this exercise, we will encounter sets in two different ways: (A) sets as the basis of formalizing various concepts (as a tool), and (B) sets as an object of study. The appearance of sets as in (A) and (B) are referred to as *meta-level* and *object level*, respectively. As an analogy, consider the following example. Nowadays, designing computer components such as CPU (Central Processing Units) heavily depends on the use of computers. Then, computers are involved both at the object level (computers/components that are being manufactured) and at the meta-level (computers used for the manufacturing process).

Let us consider a set (of sets) $A = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$ and the usual set union operation '$\cup$'. We know that $\{1\} \cup \{2\} = \{1, 2\}$, $\emptyset \cup \{2\} = \{2\}$, etc. If we combine A and '$\cup$', we should be able to construct a structure as follows:

$\mathbf{Set_1} = (A, \cup)$

- $A = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$
- '$\cup$' is the usual set union operation

A. Give the type of '$\cup$'.

B. Formally define '$\cup$' using the **list notation**. Doing this completely will be tedious. So, if you can demonstrate that you understand how to do it correctly, you may abbreviate the result.

C. Identify all the properties applicable to '$\cup$' in $\mathbf{Set_1}$ (e.g., closed, associative, commutative, existence of an identity element, existence of inverse, surjective, injective, bijective, (ir)reflexive, (anti)symmetric, transitive).

D. Among the structures discussed in class, which one would be most similar to $\mathbf{Set_1}$?

Next, consider another structure $\mathbf{Set_2}$ defined as follows:

$\mathbf{Set_2} = (A, \cap)$

- $A = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$ [same as in $\mathbf{Set_1}$]
- '$\cap$' is the usual set intersection operation

E. List the differences and similarities between $\mathbf{Set_1}$ and $\mathbf{Set_2}$.

F. Suppose that you are given a structure $\mathbf{X} = (X, \lozenge)$ with the following conditions (logical statements, informally):

- $|X| = 4$
- '$\lozenge$' is a closed, associative, commutative operation.
- There is an identity element in X.
- There are no inverses.

You are also told that **X** is "basically the same" as $\mathbf{Set_1}$, except for the use of the symbols. Would **X** be "basically the same" as $\mathbf{Set_2}$ as well (except for the use of the symbols)? Explain.

Exercise 4: FOL Review

Note: This exercise is *optional*. However, unless you are very confident about FOL formulae, you should probably do it.

Even without explicit information, we can still infer the basic information about symbols used in a FOL wff (in most cases). For example, given a wff "$\forall a\ \exists b\ (c(a) \vee d(b) = e)$", we can analyze symbols *a* through *e* as follows:

- *a* and *b* are variables (representing individuals) because they are quantified.
- *c* is a unary predicate/relation because it takes one input and evaluates to true/false. Note that '$\vee$' needs truth values on both sides.
- *d* is a unary function because it takes one input and returns an element, not a truth value.
  - Note that '=' needs elements on both sides.
  - The interpretation $(c(a) \vee d(b)) = e$ would not work because $c(a) \vee d(b)$ returns a truth value and cannot be identified with an individual using '='.
- *e* is a constant (0-ary function) because it takes no input and returns some element. Note that it is not a variable because in this course, all variables must be quantified.

A. Do an analogous analysis for the wff: $\forall f\ (\neg g(f))$

B. Do an analogous analysis for the wff: $\exists h\ \forall i\ (h = j \rightarrow k(m(h), m(i)))$

C. The following formula is *not* well-formed: $\exists n\ (p(p(n)) \wedge n = p(n))$. That is, there is no way to analyze the symbols *n* and/or *p* correctly. Explain why.

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