Review

Properties of Relations/Functions

Check properties *only for applicable cases*

- Relation _ × _
  - $A \times A$ ⇒ Reflexivity, symmetry, transitivity
- Function _ → _
  - Closedness, sujectivity, injectivity, bijectivity
  - $A \times A \to A$ ⇒ associativity, existence of an identity element, existence of inverses, commutativity
    - If negation also available ⇒ De Morgan
    - If two functions available ⇒ Distributivity

Bool = (B, &, or, ', 0, 1)

Digital circuits ⇔	Propositional logic ⇔	Set theory	$\|B\|$
{0, 1}	{**F**, **T**}	{∅, {1}}	2
			3
{00, 01, 10, 11}	{**F**, **Huh**, **Wow**, **T**}	{∅, {1}, {2}, {1, 2}}	4
			5
$\{0^n, ..., 1^n\}$ Modern computer **architecture** n-bit (2^n-bit)	{**F**, ..., ♥, ..., **T**} Knowledge representation (**AI**)	{∅, ..., {1, ..., n}} **Math**	2^n

2^∞?
BA vs. non-BA?
"Shape" of BA?

Power Set

- Power set of a non-empty set S, $P(S)$: The set of all the subset of S
  - Formally, $P(S) = \{X \mid X \subseteq S\}$
- Examples
  - $P(\{1\}) = \{\varnothing, \{1\}\}$
  - $P(\{1, 2\}) = \{\varnothing, \{1\}, \{2\}, \{1, 2\}\}$
  - $P(\{1, 2, 3\}) = \{\varnothing, \{1\}, \{2\}, \{3\}, ..., \{1, 2, 3\}\}$

Don't forget
- ∅
- The set itself

Group Exercise 1

Compute the following:

A. $P(\{a, b\})$
B. $P(\{\{a\}\})$
C. $P(\varnothing)$
D. $P(\{\varnothing\})$

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Don't forget
- ∅
- The set itself

Boolean Algebra

- Alternative characterization: Collection of structures that are isomorphic to ($P(S)$, ∩, ∪, ') for some non-empty set S
- This guarantees that all the properties are satisfied.

Why limited to power sets?

C3: Ordered Structures

Today

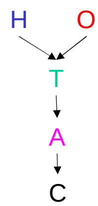
- Understand ordered structures ~ visualization
- Understand the connection between operational and relational structures
- Take-home exercises
  - House construction, Electronic map (revisited)

Section 1

CABG (Coronary Artery Bypass Graft)

Procedure

1. Opening the chest
2. Harvesting vessels (in the leg)
3. Transitioning to bypass
4. Attaching new vessels
5. Closing the chest

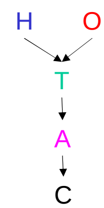


Group Exercise 2

- Consider a general case of all sorts of procedures where tasks are ordered (consider the relation "*no later than*")

A. Identify the involved properties and formally represent them
   A. Use relation symbol: '≤'

B. Formally define the collection of structure(s) **Procedure**



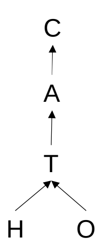
Specification of Procedure-like Structures

- Properties of '≤'
  - Reflexive
  - Antisymmetric
  - Transitive
- **Procedure** = (*Tasks*, ≤)
  - *Tasks* = the set of tasks No further information available
  - '≤': *Tasks* × *Tasks*

Poset (Partially-Ordered Set)

- **PO** = (A, ≤)
  - '≤' is a binary relation in $A \times A$.
  - '≤' is reflexive.
  - '≤' is antisymmetric.
  - '≤' is transitive.

Hasse Diagram



- Standard schematic representation for posets
- $X \leq Y$: Y is placed at least as high as X (e.g., H ≤ T or H no later than T)
- $X \leq X$: *not* shown
- $X \leq Y$ and $Y \leq Z$: $X \leq Z$ *not* shown
- I.e., reflexivity/transitivity not shown ⇔ Assumed for posets

Caution: vertical position

Group Exercise 3

- Consider a structure **X** = (S, R)

  $S = \{1, 2, 3, 4\}$

  $R = \{(1, 1), (2, 2), (3, 3), (4, 4), (1, 2), (2, 3), (1, 3), (4, 3)\}$

A. Draw a schematic representing *all* the elements of the relation

B. Is **X** a poset? If yes, draw a Hasse diagram. If no, explain.

Poset Variations

- Poset that is ordered linearly (linear order)
- Structure with the greatest and least elements but not linear
- Structure that would model "set theory"
- *Not* a poset, but similar: *Irreflexive*, antisymmetric, transitive (strict order)

How to specify these formally?

Section Summary

- Relational structures
  - Single set
  - Relation(s)
- Classification

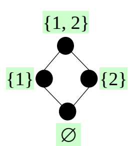
		Ref	Sym	Trans	Linearity	Max/Min
Strict order		Irref	Anti	Y	either	either
Poset	in general	Ref	Anti	Y	either	either
	Linear order	Ref	Anti	Y	Y	either
	Bounded	Ref	Anti	Y	either	Y

Section 2

Operational vs. Relational

- $\mathbf{Set_4} = (\{\emptyset, \{1\}, \{2\}, \{1, 2\}\}, \cap, \cup, ')$
- $\mathbf{Set_5} = (\{\emptyset, \{1\}, \{2\}, \{1, 2\}\}, \subseteq)$

Hasse diagram of $\mathbf{Set_5}$



Equivalence:
1. $x \subseteq y$
2. $x \cap y = x$
3. $x \cup y = y$

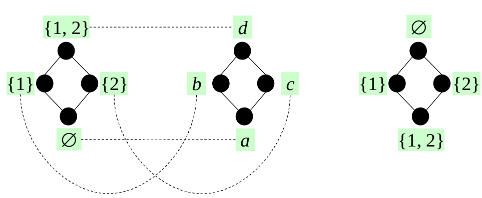
Visualize Boolean Algebra

- Hasse diagram: useful to visualize structures
- Boolean Algebra: Special collection of structures where we can move between operational and relational representations
- Consequence: Boolean Algebra can be seen through Hasse diagrams.

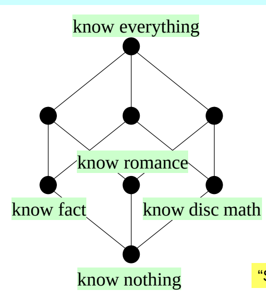
Set Operation, Revisited

Equivalence:
1. $x \subseteq y$
2. $x \cap y = x$
3. $x \cup y = y$

$\mathbf{Set1} = (A, \cup)$ $\mathbf{X} = (X, \lozenge)$ $\mathbf{Set2} = (A, \cap)$



Knowledge Representation

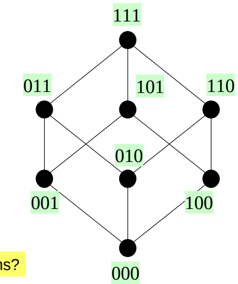


"Shape" of BA?

Boolean Algebra

- Yet another characterization: multiple-dimension lattices
  - Technically, lattice is a collection of structures that lie between Boolean Algebra and posets

3-bit Computer Architecture

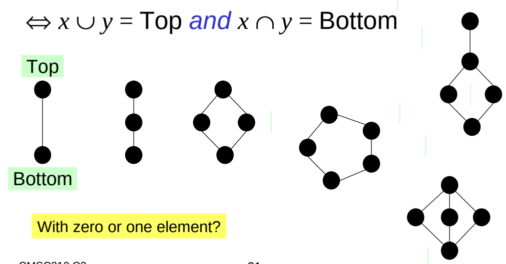


Logic variations?

BA vs. non-BA?

Complement, Revisited

- x and y are complement
  $\Leftrightarrow x \cup y =$ Top *and* $x \cap y =$ Bottom



With zero or one element?

Section Summary

- Boolean algebra corresponds to a special case of poset, where operational and relational structures can represent equivalent ideas.

	Operational	Relational
Boolean algebra e.g., digital circuits, propositional logic, set theory	$(A, \&, \text{or}, ')$ $(A, \wedge, \vee, \neg)$ $(A, \cap, \cup, ')$ other conditions	$(A, \leq)$ $(A, \subseteq)$ greatest/least distributed complemented
Poset e.g., task scheduling	N/A	$(A, \leq)$

More specific

Summary Exercise

- Can you visualize Boolean Algebra of various sorts?
- Did your idea about Boolean Algebra change after this class?
- Questions/Comments/Suggestions