

Review

Properties of Relations/Functions

Check properties *only for applicable cases*

- **Relation** $_ \times _$
 - $A \times A \Rightarrow$ Reflexivity, symmetry, transitivity
- **Function** $_ \rightarrow _$
 - Closedness, surjectivity, injectivity, bijectivity
 - $A \times A \rightarrow A \Rightarrow$ associativity, existence of an identity element, existence of inverses, commutativity
 - If **negation also available** \Rightarrow De Morgan
 - If **two functions available** \Rightarrow Distributivity

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Bool = (B, &, or, ', 0, 1)

Digital circuits	Propositional logic	Set theory	B
{0, 1}	{F, T}	{ \emptyset , {1}}	2
			3
{00, 01, 10, 11}	{F, Huh, Wow, T}	{ \emptyset , {1}, {2}, {1, 2}}	4
			5
{ $0^n, \dots, 1^n$ Modern computer architecture n -bit (2^n -bit)	{F, ..., ♥, ..., T} Knowledge representation (AI)	{ $\emptyset, \dots, \{1, \dots, n\}$ Math	2^n
		2 ⁿ ? BA vs. non-BA? "Shape" of BA?	

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Power Set

- Power set of a non-empty set S , $P(S)$: The set of all the subset of S
 - Formally, $P(S) = \{X \mid X \subseteq S\}$
- Examples
 - $P(\{1\}) = \{\emptyset, \{1\}\}$
 - $P(\{1, 2\}) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$
 - $P(\{1, 2, 3\}) = \{\emptyset, \{1\}, \{2\}, \{3\}, \dots, \{1, 2, 3\}\}$

Don't forget

- \emptyset
- The set itself

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Group Exercise 1

Compute the following:

A. $P(\{a, b\})$

B. $P(\{\{a\}\})$

C. $P(\emptyset)$

D. $P(\{\emptyset\})$

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Don't forget

- \emptyset
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Boolean Algebra

- **Alternative characterization:** Collection of structures that are isomorphic to $(P(S), \cap, \cup, ')$ for some non-empty set S
- This guarantees that all the properties are satisfied.

Why limited to power sets?

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C3: Ordered Structures

Today

- Understand ordered structures ~ visualization
- Understand the connection between operational and relational structures
- Take-home exercises
 - House construction, Electronic map (revisited)

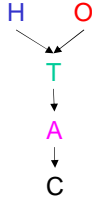
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Section 1

CABG (Coronary Artery Bypass Graft)

Procedure

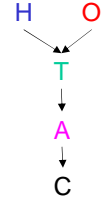
1. Opening the chest
2. Harvesting vessels (in the leg)
3. Transitioning to bypass
4. Attaching new vessels
5. Closing the chest



Group Exercise 2

- Consider a general case of all sorts of procedures where tasks are **ordered** (consider the relation “no later than”)

- A. Identify the involved properties and formally represent them
 - A. Use relation symbol: ‘ \leq ’
- B. Formally define the collection of structure(s) **Procedure**



Specification of Procedure-like Structures

- Properties of ‘ \leq ’
 - Reflexive
 - Antisymmetric
 - Transitive
- **Procedure** = $(Tasks, \leq)$
 - $Tasks$ = the set of tasks No further information available
 - ‘ \leq ’: $Tasks \times Tasks$

Poset (Partially-Ordered Set)

- **PO** = (A, \leq)
 - ‘ \leq ’ is a binary relation in $A \times A$.
 - ‘ \leq ’ is reflexive.
 - ‘ \leq ’ is antisymmetric.
 - ‘ \leq ’ is transitive.

Hasse Diagram

- Standard schematic representation for posets
- $X \leq Y$: Y is placed at least as high as X (e.g., $H \leq T$ or H no later than T)
- $X \leq X$: **not** shown
- $X \leq Y$ and $Y \leq Z$: $X \leq Z$ **not** shown
- I.e., **reflexivity/transitivity** not shown \Leftrightarrow Assumed for posets



Caution: vertical position

Group Exercise 3

- Consider a structure $\mathbf{X} = (S, R)$
 - $S = \{1, 2, 3, 4\}$
 - $R = \{(1, 1), (2, 2), (3, 3), (4, 4), (1, 2), (2, 3), (1, 3), (4, 3)\}$
- A. Draw a schematic representing **all** the elements of the relation
- B. Is \mathbf{X} a poset? If yes, draw a Hasse diagram. If no, explain.

Poset Variations

- Poset that is ordered linearly (**linear order**)
- Structure with the greatest and least elements but not linear
- Structure that would model “set theory”
- **Not** a poset, but similar: *Irreflexive*, antisymmetric, transitive (strict order)

How to specify these formally?

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Section Summary

- Relational structures
 - Single set
 - Relation(s)
- Classification

	Ref	Sym	Trans	Linearity	Max/Min	
Strict order	Irref	Anti	Y	either	either	
Poset	in general	Ref	Anti	Y	either	either
	Linear order	Ref	Anti	Y	Y	either
	Bounded	Ref	Anti	Y	either	Y

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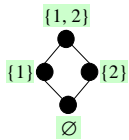
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Section 2

Operational vs. Relational

- $\text{Set}_4 = (\{\emptyset, \{1\}, \{2\}, \{1, 2\}\}, \cap, \cup, ')$
- $\text{Set}_5 = (\{\emptyset, \{1\}, \{2\}, \{1, 2\}\}, \subseteq)$

Hasse diagram of Set_5



Equivalence:

1. $x \subseteq y$
2. $x \cap y = x$
3. $x \cup y = y$

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Visualize Boolean Algebra

- **Hasse diagram**: useful to visualize structures
- **Boolean Algebra**: Special collection of structures where we can move between operational and relational representations
- Consequence: Boolean Algebra can be seen through Hasse diagrams.

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Set Operation, Revisited

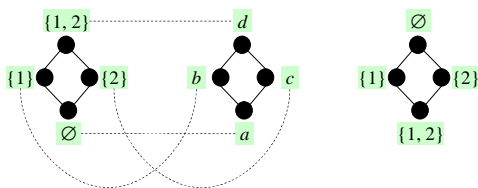
Equivalence:

1. $x \subseteq y$
2. $x \cap y = x$
3. $x \cup y = y$

Set1 = (A, ∪)

X = (X, ∩)

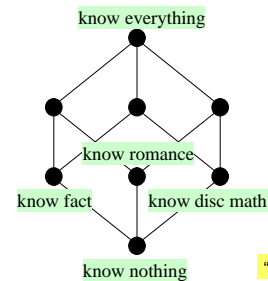
Set2 = (A, ∩)



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Knowledge Representation



“Shape” of BA?

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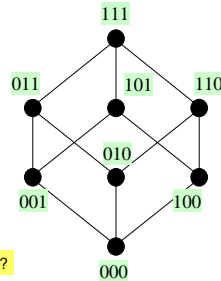
Boolean Algebra

- Yet another characterization: multiple-dimension lattices
 - Technically, lattice is a collection of structures that lie between Boolean Algebra and posets

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3-bit Computer Architecture



Logic variations?

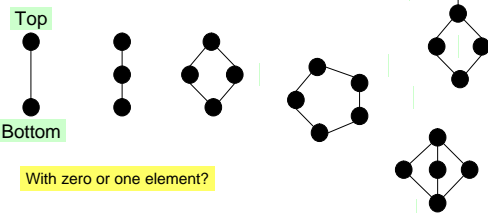
BA vs. non-BA?

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Complement, Revisited

- x and y are complement
 - $\Leftrightarrow x \cup y = \text{Top}$ and $x \cap y = \text{Bottom}$



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Section Summary

- Boolean algebra corresponds to a special case of poset, where operational and relational structures can represent equivalent ideas.

	Operational	Relational
Boolean algebra e.g., digital circuits, propositional logic, set theory	$(A, \&, \text{or}, ')$ (A, \wedge, \vee, \neg) $(A, \cap, \cup, ')$ other conditions	(A, \leq) (A, \subseteq) greatest/least distributed complemented
Poset e.g., task scheduling	N/A	(A, \leq)

More specific

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Summary Exercise

- Can you visualize Boolean Algebra of various sorts?
- Did your idea about Boolean Algebra change after this class?
- Questions/Comments/Suggestions

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