

Defining % (remainder)?

- ‘%’: $\mathbf{N} \times \mathbf{N} \rightarrow \mathbf{N}$
- ‘%’ = $\{((0, 1), 0), ((1, 1), 0), \dots, ((0, 2), 0), ((1, 2), 1), \dots\}$ **infinitely many members!**
- ‘%’ = $\{((x, y), z) \mid z = x \% y\}$ **circular!**
- ‘%’ = $\{((x, y), z) \mid x = y \times n + z \text{ for some } n, \text{ where } x, y, n, z \in \mathbf{N} \text{ and } z < y\}$

Then, how to define ‘+’?

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1

How to define these?

- Addition
- Natural numbers
- Strings
- Cities which can be traveled from Trenton by driving
- All of your ancestors **Are they all human?**

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2

Review: Set Definition

- Finite sets
 - Can be defined using the list notation
- Infinite sets **Does infinity exist? Useful?**
 - Cannot be defined using the list notation
 - Could be defined using the predicate notation
 - Must avoid circular definitions
 - Could be defined in terms of more primitive sets

Most primitive infinite set?

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3

Hypothesis

- If we can define the set of natural numbers, we can define all sorts of infinite sets.
- Consequence: You can define even **PingPong** (including ‘%’) completely formally.

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4

Module D

- Foundation
- Most abstract **Moved toward the end**
- Intellectual challenge ~ problem solving skills ~ mathematical reasoning skills
- Tools to convince others (who understand logic and mathematics, e.g., people in science and engineering)

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5

Unit D1: Sets

Today

- Defining **natural numbers**
- Understand and use **inductive definition** of a set
- Take-home exercises
 - Web pages, Foreign policy

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6

Defining Natural Numbers

- Approach: Specification using logical statements
- To specify a structure that would contain \mathbf{N} (the set of natural numbers) as the primary set (called *carrier*)

Will make several attempts

Group Exercise 1.1

- Consider a collection of structures $\mathbf{Nat}_1 = (A, \leq, 0)$ specified by the following:
 1. $\exists x (x = 0)$
 2. $\forall x \exists y (x < y)$ *$x < y$ is abbreviation for $x \leq y \wedge x \neq y$*
- Identify multiple instances of \mathbf{Nat}_1 that would satisfy the conditions
 - Number of elements?
 - Represent schematically

Group Exercise 1.2

- Consider a collection of structures $\mathbf{Obj}_1 = (O, \text{noHigherThan}, \text{Bottom})$ specified by the following:
 1. $\exists x (x = \text{Bottom})$ *$x \text{ below } y$ is abbreviation for $x \text{ noHigherThan } y \wedge x \neq y$*
 2. $\forall x \exists y (x \text{ below } y) \Leftrightarrow \forall x \exists y (y \text{ above } x)$
- Identify multiple instances of \mathbf{Obj}_1 that would satisfy the conditions
 - Number of elements?
 - Represent schematically

Group Exercise 1.3

- Consider a collection of structures $\mathbf{Plot}_1 = (O, \preceq, \bullet^*)$ specified by the following:
 1. $\exists x (x = \bullet^*)$ *$x \preceq y$ is abbreviation for $x \prec y \wedge x \neq y$*
 2. $\forall x \exists y (x \preceq y)$
- Identify multiple instances of \mathbf{Plot}_1 that would satisfy the conditions
 - Number of elements?
 - Represent schematically

More Things to Check

- Linear?
- Least element?
- Hasse diagram? I.e., poset?
- Smallest structure?

Group Exercise 2

- Consider a collection of structures $\mathbf{Nat}_2 = (A, \text{succ}, 0)$ specified by the following:
 1. $\exists x (x = 0)$ *succ for successor, but do not be trapped by assumptions*
 2. $\forall x \exists y (y = \text{succ}(x))$
 - Identify multiple instances of \mathbf{Nat}_2 that would satisfy the conditions
 - Number of elements?
 - Represent schematically
- Think flexibly

Conditions for Natural Numbers

- Linearity ~ special case of poset
- Least element
- Intended meaning for ' \leq ' and *succ*

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13

Specifying Natural Numbers

- $\text{Nat}_3 = (A, \leq, \text{succ}, 0)$
- Conditions
 - Poset
 - Linearity
 - Least element
 - Definition of *succ*

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14

$\text{Nat}_3 = (A, \leq, \text{succ}, 0)$

- Poset
 - $\forall x (x \leq x)$
 - $\forall x \forall y ((x \leq y \wedge y \leq x) \rightarrow x = y)$
 - $\forall x \forall y \forall z ((x < y \wedge y < z) \rightarrow x < z)$
- Linearity: $\forall x \forall y (x \leq y \vee y \leq x)$
- Least element: $\forall x (0 \leq x)$
- Definition of *succ*
 - $\forall x \exists y (x \neq 0 \rightarrow x = \text{succ}(y))$
 - $\forall x \forall y (x < \text{succ}(y) \leftrightarrow x \leq y)$

succ for successor
 $x < y$ is abbreviation for $x \leq y \wedge x \neq y$

Number Theory (Math)

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15

Digression: Hidden Problem

- The conditions for Nat_3 can be satisfied by many unintended structures.
- In fact, there are infinite many of them.

Model Theory (Mathematical Logic)

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16

Section Summary

- It is possible to define the set of natural numbers using the logic-structure connection.
- Disadvantages
 - Complicated
 - Still unintended structures
- Need for a convenient pattern to define infinite sets

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17

Strings

- String** = $(S, +, \varepsilon)$
- S : set of all strings
 - How to define A formally?

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18

Inductive Definition of \mathbf{N}

- **Base case:** $0 \in \mathbf{N}$
- **Induction step:** If $x \in \mathbf{N}$, $\text{succ}(x) \in \mathbf{N}$
- **Exclusion clause:** Nothing else is in \mathbf{N} .

Schematic

- Abbreviation/alternatives
 - Inductive definition: recursive definition
 - Base case: Base, basis
 - Exclusion clause: Exclusion

In FOL?

Essence of Inductive Definition

- Base case
 - Specification of starting members (finite number)
- Induction step
 - Specification of how to grow the set (potentially infinitely)
- Exclusion clause
 - Specification of how to stop the growth

Integers

- Define the set of integers: I
 - Assume: $\text{prev} = \text{succ}^{-1}$ (i.e., the inverse)
- Base:
- Induction step:
- Exclusion:

Strings (Group Exercise 3)

- Define the set of strings: S
 - $\Sigma = \{a, b, c, \dots, z\}$
 - $\text{attachChar}: S \times \Sigma \rightarrow S$
- Base:
- Induction step:
- Exclusion:

Cf. definition of strings using a regular expression

Ancestors (1)

- Define the set of your ancestors: A
 - $\text{isParentOf}: A \times A$
- Base: $\text{you} \in A$
- Induction step:
- Exclusion: Nothing else is in A .

Are we all brothers/sisters?

Ancestors (2)

- Define the set of your ancestors: A
 - $\text{parents}: A \rightarrow ____$ (returns all the parents)
- Base: $\text{you} \in A$
- Induction step:
- Exclusion: Nothing else is in A .

Cities

- Define the set of cities drivable from Trenton: C
 - *drivableFromTo*: $C \times C$
- Base: $Trenton \in C$
- Induction step:
- Exclusion: Nothing else is in C .

Infinite?
Undirected graph?

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25

Section Summary

- Inductive definition is a convenient way to precisely and concisely define potentially infinite sets.
 - Cf. the definition of Nat_3 in Section 1

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26

Summary Exercise

- Can you define all the possible chess positions inductively? [no need to actually do this; just explain whether you can do it]
- Questions/Comments/Suggestions

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27