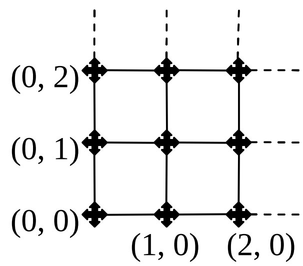
Unit D2: Relations, 11/21/03

Exercise 1: Space Stations

Imagine a huge network of space stations (✥) organized on a two-dimensional (2D) space, as shown below. Stations have direct communication links to the neighboring stations (solid line). Through possibly multiple connections, every station can communicate with any station in the network. Although the leftmost and the bottom boundaries of the 2D space are bounded as in the diagram, we assume infinitely many connections upward and rightward, considering the huge size. The stations can be identified as a pair of natural numbers, as shown in the figure.



A. Formally define the relation *C*, with the meaning "can communicate with" (between space stations). You may use **N** (the set of natural numbers) in your definition.

   Hint: There are different ways of answering this question. One answer is very short.

B. Prove that *C* is an equivalence relation.

   Note: Try to follow the proof pattern shown on slides. However, you do not need to use lemmas if you feel more comfortable proving without them.

C. Due to the sudden appearance of a black hole, we are told that two distinct stations (x_1, y_1) and (x_2, y_2) cannot communicate at all (not even indirectly). Is *C* still necessarily an equivalence relation? Explain carefully by also referring to equivalence classes and partition.

Exercise 2: Bogus Proofs and Paradoxes

There is a proverb: "logic is a way to go wrong with confidence." Occasionally, this seems quite true (although we won't be able to prove or disprove this proverb). To understand proofs well, we will examine some bogus proofs and paradoxes in this exercise.

A. There is a problem in the following attempted proof (by contradiction). Explain the problem.

   Statement to prove (from no hypotheses): "God exists."

   Proof attempt
   1. The word *God* exists. [any dictionary]
   2. | Suppose that God does not exists. [hyp to reject]

3. | Then, there must not be the word *God*. [2.]
4. | A contradiction. [1., 3.]
5. Therefore, God exists. [proof by contradiction: 2.-4.]

Note: Naturally, the existence of a bogus proof does not disprove the statement.

B. (i) Analyze the truth value of the following statement: “This sentence is false.” (ii) Speculate whether or not this statement can be formally represented in First-Order Logic (FOL). (iii) If it is possible to represent this statement in FOL, what can you conclude?

Hint: For (i), consider the following two cases: the entire statement is (a) true and (b) false.

Note: (ii) and (iii) are challenging. Try your best.

C. Create your own example of either (i) a bogus proof or (ii) a logical paradox. The example must be *unique to you* (i.e., different from class examples/exercises or shared by your collaborators).

Alternative: If it is difficult, you may do a web/library search and find such an example. In this case, clearly indicate the source of the information (e.g., URL).

Note: In case you were unable to find one despite your effort, explain what you did.

<End>