Unit D4: Logic, 12/2/03

Sample solutions to these exercises will be posted in the afternoon of the due date. But do these problems without seeing the solutions. Being able to understand provided solutions is *completely* different from being able to come up with a solution. Even an incomplete or incorrect answer of your own is better than understanding someone else' answer. These exercises contain some earlier materials so that you can do some practice for the final.

Exercise 1: Highway Design

Let us model a country with n cities and highway routes among the cities, as an **undirected** graph, **G** = (C, R). Here, C is a non-empty set of cities (vertices) with $|C| = n$, and R is a set of routes (edges).

A. Suppose that the undirected graph **G** is *connected*, but ***not*** a *tree*. What would be the minimal $|R|$ that satisfies this condition? Explain.

B. You learned that there are at least two cities that are not connected by a highway route. You are also told that the reflexive closure of R is an equivalence relation. What would be the largest $|[x]|$ (based on this equivalence relation) where x is some element in C? Explain.

C. Suppose that there is a route between every pair of cities. Then, the total number of routes (i.e., $|R|$) is equal to $(n - 1) \times n / 2$, for any $n \geq 1$. Fill in the proof by mathematical induction given below.

Answer:

A. $2n$. To be connected, there must be at least $2(n - 1)$ routes (counting both directions separately, as an undirected graph needs to be symmetric). To avoid being a tree, there must be at least one cycle. Thus, we need an additional pair of routes, resulting in $2n$ routes.

B. $n - 1$. Since there are at least two cities that are not connected by a route, there must be at least two equivalence classes. The largest such equivalence class should include $n - 1$ cities.

C.

Main hypothesis: There is a route between every pair of cities.

Main conclusion: $|R| = (n - 1) \times n / 2$

Base case ($|V| = n = 1$ since V must be non-empty): $|R| = (n - 1) \times n / 2 = 0$ (i.e., no connection needed if there is only one city)

Induction step

Induction hypothesis: The formula is correct for n, i.e., for $|V| = n$, the total number of routes is $(n - 1) \times n / 2$.

Conclusion: The formula is correct also for $n + 1$, i.e., for $|V| = n + 1$, $|R| = n \times (n + 1) / 2$.

Proof (of the induction step):

1. By adding another city to the existing ones, we will need n more routes to connect the new city to every other cities. [Induction hyp. and Main hyp.]
2. We already have the formula $(n - 1) \times n / 2$ for the existing n cities. [Induction hypothesis]
3. Then, $(n - 1) \times n / 2 + n = n \times (n + 1) / 2$, which is the correct formula for $n + 1$. [Arithmetic: 1., 2.]

By Mathematical Induction, the **main** conclusion holds.

Exercise 2: Teriyaki Boy

"Teriyaki Boy" at a local shopping mall offers a variety of sushi platters. To satisfy an even broader customer base, they are considering to offer extravagant twenty (20) sushi kinds: abalone (*a*), bonito (*b*), crab (*c*), dover sole (*d*), eel (*e*), flounder (*f*), giant clam (*g*), ..., and tuna (*t*).

The Japanese are good at importing useful ideas from their neighbors (e.g., the Chinese writing system). Teriyaki Boy might adopt another Chinese system of numbering lunch menus from #1 through #*n*.

Note: In your answers, you may use the notations *P* (permutation), *C* (combination), '!' (factorial), and constants. You should ***not*** compute actual numeric values.

A. Suppose that lunch menus include 4 pieces of sushi of exactly 3 different kinds. How can you compute the value of n (the last lunch menu number)?

   Hint: Consider choosing 3 kinds and then finding the distribution of the items.

B. Presentation is an important part of the Japanese cuisine. Suppose that a customer ordered a special party platter that includes 4 pieces of eel (*e*), 3 pieces of flounder (*f*), and 2 pieces of giant clam (*g*). All the sushi pieces must be arranged in a single, non-circular row. In how many different ways can a chef present this party platter?

   Note: Assume that sushi plates have a **asymmetrical** shape. That is, presentations of sushi items [*e*, *e*, *e*, *f*, *g*, *f*, *e*, *f*, *g*] and [*g*, *f*, *e*, *f*, *g*, *f*, *e*, *e*, *e*] **must** be distinguished.

The function $f(n) = 2^n$ grows really fast as the input grows. For example, see the following cases: $f(1) = 2$, $f(8) = 256$, $f(16) = 65536$, $f(32) = 4294967296$, and $f(64) = 18446744073709551616$. However, $g(n) = n!$ is even worse, e.g., $g(16) = 20922789888000 >> f(16)$. That's why you are not asked to compute numeric values in the above questions.

C. To convince yourself, prove the following equation, *using mathematical induction*: $2^n < n!$, for any natural number $n \geq 4$. Follow the Mathematical Induction proof pattern.

Answer:
A. $C(20, 3) \times C(3, 1)$ First, the possibilities of choosing 3 kinds from 20 is $C(20, 3)$. Suppose we chose kinds x, y, and z. To include 4 pieces, we will need to have two items of just one kind, e.g., *xxyz*, *xyyz*, etc.

B. $P(9, 9)/(P(4, 4) \times P(3, 3) \times P(2, 2)) = 9!/(4! \times 3! \times 2!)$ First, there are $P(9, 9)$ ways to arrange 9 pieces. However, since there are same kinds, we must eliminate the effect of different ordering of the same kind, e.g., 4 pieces of eel, by dividing by $P(4, 4)$.

C.
Main hypothesis: n is a natural number and $n \geq 4$
Main conclusion: $2^n < n!$

Base case ($n = 4$): $4^2 = 16 < 4! = 24$
Induction step
Induction hypothesis: $2^n < n!$
Conclusion: $2^{(n+1)} < (n + 1)!$
Proof (of the induction step):

1. $2^{(n+1)} = 2 \times 2^n$ [Def. power, applied to the LHS of Conclusion]
2. $2 \times 2^n < 2 \times n!$ [Ind. hyp., multiplied by 2]
3. $2 < n + 1$ [Main hyp.]
4. $2 \times n! < (n + 1) \times n!$ [3., multiplied by $n!$]
5. $(n + 1) \times n! = (n + 1)!$ [Def. '!': 4.]
6. $2^{(n+1)} < (n + 1)!$ [Transitivity of equality/inequality: 1., 2., 4., 5.]

By Mathematical Induction, the **main** conclusion holds.

Disclaimer: All the names in this exercise are fictitious. Resemblance to any real name is coincidental.

Exercise 3: Program Analysis

The performance of a computer program is often described in terms of a function of the input size n (e.g., the number of database entries, the length of the input string). Suppose that the behavior of some program (Program 1) can be characterized by $f(n) = n^2$ and that of another (Program 2) by $g(n) = 2^n$. These two programs are compared below for some small input sizes (a variety of n values). Between Programs 1 and 2, a smaller value means a better performance (less time spent).

	Program 1	Program 2
1	1	2
2	4	4
3	9	8
4	16	16
5	25	32
|	|	|
i	i^2	2^i
|	|	|

For $n \leq 5$, neither program performs consistently better than the other. However, for the size $n \geq 5$, Program 1 consistently outperforms Program 2. This property can be represented as the following formula: $n^2 < 2^n$ for any $n \geq 5$ [**Theorem**]. We will justify this formula using mathematical induction.

A. In the justification of **Theorem**, we will use a lemma (preliminary theorem): $2n + 1 < n^2$ for any $n \geq 5$ [**Lemma**]. Justify this lemma *using mathematical induction*. Follow the Mathematical Induction proof pattern.

B. Justify **Theorem** *using mathematical induction* and **Lemma**. Follow the Mathematical Induction proof pattern.

Answer:

A.

Main hypothesis: $n \geq 5$

Lemma: $2n + 1 < n^2$

Base case ($n = 5$): $2 \times 5 + 1 = 11 < 5^2 = 25$

Induction step

Induction hypothesis: **Lemma** is correct for n, i.e., $2n + 1 < n^2$

Conclusion: **Lemma** is correct also for $n + 1$, i.e., $2(n + 1) + 1 < (n + 1)^2$

Proof (of the induction step):

1. $2(n + 1) + 1 = 2n + 3 = (2n + 1) + 2$ [Arithmetic: LHS of Conclusion]
2. $(n + 1)^2 = n^2 + 2n + 1$ [Arithmetic: RHS of Conclusion]
3. $2n + 1 < n^2$ [Ind. hyp.]
4. $(2n + 1) + 2 < n^2 + 2$ [3., 2 added]
5. $2 < 2n + 1$ [Main hyp.]
6. $n^2 + 2 < n^2 + 2n + 1$ [5., n^2 add]
7. $n^2 + 2n + 1 = (n + 1)^2$ [Arithmetic: 6.]
8. $2(n + 1) + 1 < n^2 + 2n + 1$ [Transitivity of equality/inequality: 1., 4., 6.]

By Mathematical Induction, **Lemma** holds for all natural numbers $n \geq 5$.

B.

Main hypothesis: $n \geq 5$

Theorem: $n^2 < 2^n$

Base case ($n = 5$): $5^2 = 25 < 2^5 = 32$

Induction step

Induction hypothesis: **Theorem** is correct for n, i.e., $n^2 < 2^n$

Conclusion: **Theorem** is correct also for $n + 1$, i.e., $(n + 1)^2 < 2^{n+1}$

Proof (of the induction step):

1. $(n + 1)^2 = n^2 + 2n + 1$ [Arithmetic: LHS of Conclusion]
2. $2n + 1 < n^2$ [**Lemma**]
3. $n^2 + 2n + 1 < n^2 + n^2$ [4., n^2 added]
4. $n^2 < 2^n$ [Ind. hyp.]

5. $n^2 + n^2 < 2^n + 2^n$ [5.]
6. $2^n + 2^n = 2^{n+1}$ [Def. power]
7. $(n + 1)^2 < 2^{n+1}$ [Transitivity of equality/inequality: 1., 3., 5., 6.]

By Mathematical Induction, **Theorem** holds for all natural numbers $n \geq 5$.

Exercise 4: Producer-Consumer [optional]

We observed that the following two mutually-recursive functions terminate regardless of the inputs. Can you prove this *using Mathematical Induction*?

$$produce(m) = \begin{cases} m & \text{if } m > 9 \\ consume(m \times 1.25) & \text{otherwise} \end{cases}$$

$$consume(m) = \begin{cases} m & \text{if } m < 1 \\ produce(m \times 0.75) & \text{otherwise} \end{cases}$$

Note: This is not really the Mathematical Induction on natural numbers. However, the same proof pattern is equally convincing.

Hint: Set up four base cases where the functions terminate immediately (2 cases) and the function terminates after one recursion (2 cases). For the proof of the induction step, use the fact $1.25 \times 0.75 = 0.9375$, and demonstrate that two recursive steps will always be the same function with a reduced input.

Answer:
Main hypothesis: None
Main conclusion: Both *produce* and *consume* terminate on any input.

Base cases

- $consume(m)$ terminates if $m < 1$.
- $consume(m)$ terminates if $m \times 0.75 > 9$, as $consume(m) = produce(m \times 0.75)$.
- $produce(m)$ terminates if $m > 9$.
- $produce(m)$ terminates if $m \times 1.25 < 1$, as $produce(m) = consume(m \times 1.25)$.

Induction step

Induction hypothesis: $consume(c)$ terminates for some $c \geq 1$.
Conclusion: $consume(d)$ and $produce(d)$ both terminate for any $d > c$.
Proof (of the induction step):
Case (a) $produce(m) = consume(m \times 1.25) = produce(m \times 1.25 \times 0.75) = produce(m \times 0.9375)$. Since $m \times 0.9375 < m$, by the induction hypothesis, $produce(m \times 0.9375)$ terminates. So does $produce(m)$.
Case (b) Analogous for $consume(m)$.

By Mathematical Induction, the main conclusion holds.

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