

Announcements

- Final Evaluation Workshop
 - Section 1: 8:00-10:00 a.m. in HH253
 - Section 2: 11:00 a.m.-1:00 p.m. in HH253
 - 4 sessions for 4 exercises
 - No mutual evaluation of Mini Projects
- Final Comprehensive Exercises
 - To be available on-line on Thursday
 - Recommended to preview and ask questions on Friday

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Related problem in FinComp

Producer-Consumer

- Condition for termination?

$$produce(m) = \begin{cases} m & \text{if } m > 9 \\ consume(m \times 1.25) & \text{otherwise} \end{cases}$$

$$consume(m) = \begin{cases} m & \text{if } m < 1 \\ produce(m \times 0.75) & \text{otherwise} \end{cases}$$

Mutual recursion

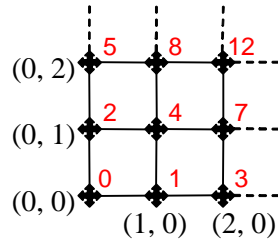
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How to prove it?

Review: Unit D2 Ex1

Labeling stations sequentially



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Related problem in FinComp

Stations	C	0	1	2	3	4	5	6	7	8
0	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
1	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
2	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
3	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
4	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
5	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
6	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
7	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
8	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓

Reflexive, symmetric, transitive? Visualize?

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Reflexive

C	0	1	2	3	4	5	6	7	8
0	✓								
1		✓							
2			✓						
3				✓					
4					✓				
5						✓			
6							✓		
7								✓	
8									✓

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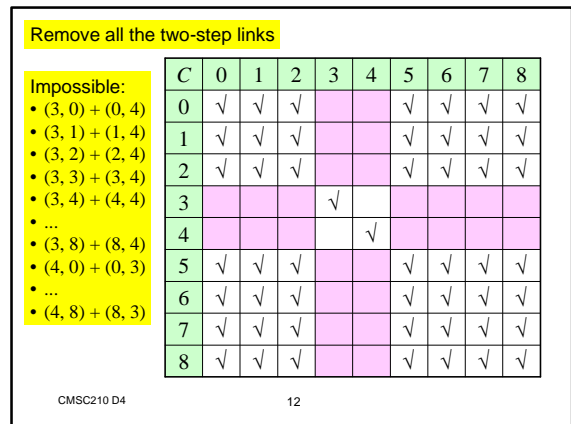
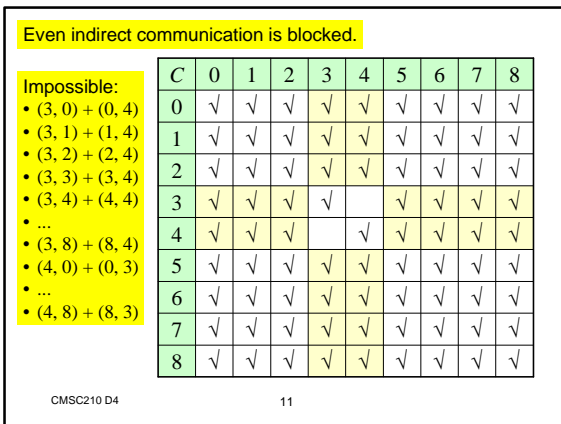
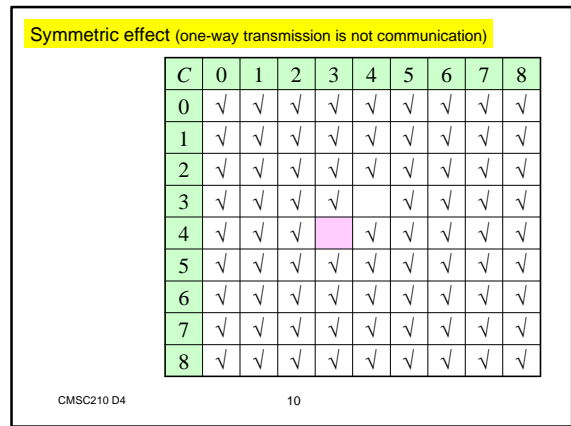
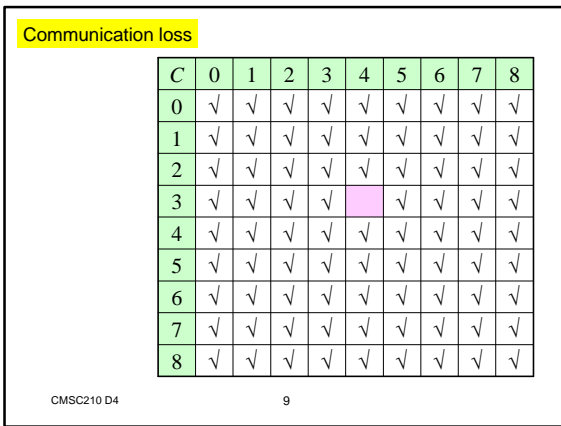
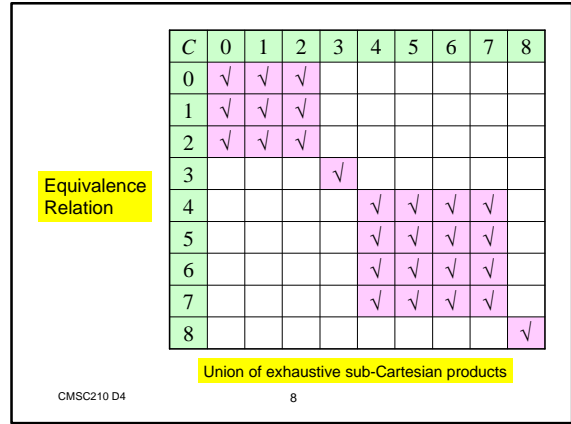
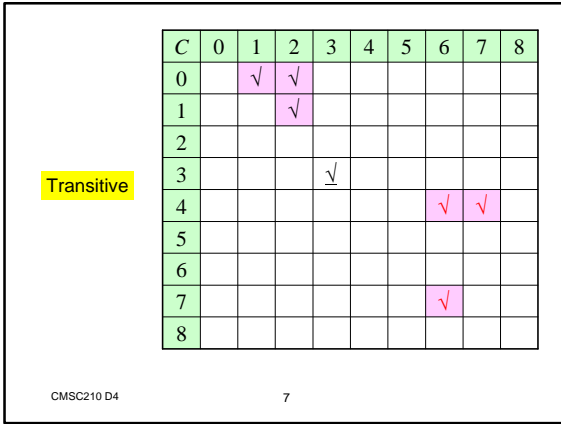
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Symmetric

C	0	1	2	3	4	5	6	7	8
0		✓							
1	✓		✓						
2		✓							
3				✓	✓			✓	
4				✓				✓	
5				✓				✓	
6					✓	✓			
7					✓			✓	
8									✓

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Rearranged: Still an equivalence relation

C	0	1	2	5	6	7	8	3	4
0	√	√	√	√	√	√	√		
1	√	√	√	√	√	√	√		
2	√	√	√	√	√	√	√		
5	√	√	√	√	√	√	√		
6	√	√	√	√	√	√	√		
7	√	√	√	√	√	√	√		
8	√	√	√	√	√	√	√		
3								√	
4									√

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Alternative: Blocking just one of the two steps?

- Impossible:
- (3, 0) + (0, 4)
 - (3, 1) + (1, 4)
 - (3, 2) + (2, 4)
 - (3, 3) + (3, 4)
 - (3, 4) + (4, 4)
 - ...
 - (3, 8) + (8, 4)
 - (4, 0) + (0, 3)
 - ...
 - (4, 8) + (8, 3)

C	0	1	2	3	4	5	6	7	8
0	√	√	√	√	√	√	√	√	√
1	√	√	√	√	√	√	√	√	√
2	√	√	√	√	√	√	√	√	√
3	√	√	√	√		√	√	√	√
4	√	√	√		√	√	√	√	√
5	√	√	√	√	√	√	√	√	√
6	√	√	√	√	√	√	√	√	√
7	√	√	√	√	√	√	√	√	√
8	√	√	√	√	√	√	√	√	√

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Remove one of the two steps: Still an equivalence relation

- Impossible:
- (3, 0) + (0, 4)
 - (3, 1) + (1, 4)
 - (3, 2) + (2, 4)
 - (3, 3) + (3, 4)
 - (3, 4) + (4, 4)
 - ...
 - (3, 8) + (8, 4)
 - (4, 0) + (0, 3)
 - ...
 - (4, 8) + (8, 3)

C	0	1	2	3	4	5	6	7	8
0	√	√	√	√					
1	√	√	√	√					
2	√	√	√	√					
3	√	√	√	√					
4					√	√	√	√	√
5					√	√	√	√	√
6					√	√	√	√	√
7					√	√	√	√	√
8					√	√	√	√	√

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Unit D5: Logic

Focus on Mathematical Induction

Today

- Understand and use **Mathematical Induction**
- Compare inductive techniques
- Understand the general meaning of **deduction** and **induction**
- Take-home exercises
 - Highway, Teriyaki Boy, Program analysis

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Section 1

Sum of Linear Series

$$\sum_{i=1}^n i = 1 + 2 + \dots + n = \frac{n \times (n + 1)}{2}$$

- Gauss' story
- Mathematical induction

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Mathematical Induction

- A widely accepted **proof technique**
- Let $\mathbf{N} = \{0, 1, 2, 3, \dots\}$
- Let $P(n)$ be a unary relation (set) on $n \in \mathbf{N}$
- If both of the following conditions hold:
 - $P(n_0)$, where $n_0 \in \mathbf{N}$ **e.g., $n_0 = 0$**
 - For every $n \geq n_0$ ($n \in \mathbf{N}$), $P(n)$ implies $P(n + 1)$
- Then, $P(n)$ is true for **every** $n \geq n_0$ ($n \in \mathbf{N}$)

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Math Induction Proof Pattern

- Main hypothesis: None
- Main conclusion: $\sum_{1 \leq i \leq n} i = n \times (n + 1) / 2$
- Base case ($n = 1$): $\sum_{1 \leq i \leq 1} i = 1 = 1 \times (1 + 1) / 2$
- Induction step
 - Induction hypothesis: $\sum_{1 \leq i \leq n} i = n \times (n + 1) / 2$
 - Conclusion: $\sum_{1 \leq i \leq (n + 1)} i = (n + 1) \times (n + 2) / 2$
 - Proof (of the induction step)
 1. $\sum_{1 \leq i \leq (n + 1)} i = \sum_{1 \leq i \leq n} i + (n + 1)$ [LHS Conclusion]
 2. $\sum_{1 \leq i \leq n} i + (n + 1) = n \times (n + 1) / 2 + (n + 1)$ [Ind. hyp.]
 3. $n \times (n + 1) / 2 + (n + 1) = (n + 1) \times (n + 2) / 2$ [Arithmetic]
 4. $\sum_{1 \leq i \leq (n + 1)} i = (n + 1) \times (n + 2) / 2$ [Transitivity eq/ineq: 1.-3.]
- By **Math Induction** (shaded), main conclusion holds.

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Pigeonhole Principle

- Theorem: If $n + 1$ letters are put in n mailboxes, then some mailbox will contain (at least) two letters.
- Proof
 - Base case ($n = 1$): The mailbox contains two letters.
 - Induction step
 - Induction hypothesis: The theorem holds for n .
 - Conclusion: **It holds for $n + 1$ as well.**
 - Proof (of the induction step)
 - Case 1 (first mailbox has 2 letters): Done.
 - Case 2 (otherwise): _____
 - By Mathematical Induction, the theorem holds.

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Group Exercise 1

- $(1 + 2 + \dots + n)^2 = 1^3 + 2^3 + \dots + n^3$
for **any** positive integer n

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Cardinality of Power Sets

- Prove that $|P(S)| = 2^{|S|}$ for **any** set S (including \emptyset)
 - $P(S)$: the power set of a set S
 - $|S|$: the cardinality of set S
- Hint: Induction on the **size** of S

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Section 2

Inductive Techniques

- Inductive definition of **sets**
 - Recursive definition of **functions**
 - Mathematical induction (a **proof** technique)
 - Common feature
 - Base case
 - Induction step
- Finite means to deal with unbounded situations (potential infinity)
- Also "Exclusion clause" for set definition

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Inductive Set Definition

Example

- wff's in propositional logic
 - Base: Propositional variables are a wff.
 - Induction step:
 - If ϕ is a wff, $(\neg\phi)$ is a wff.
 - If ϕ and ψ are wff's, $(\phi \wedge \psi)$, $(\phi \vee \psi)$, and $(\phi \rightarrow \psi)$ are a wff.
 - Exclusion: Nothing else is a wff.

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Recursive Function Definition

- Replace all the patterns of $X \rightarrow Y$ to $\neg X \vee Y$

$$f(X) = \begin{cases} X & \text{if } X \text{ is a propositional variable} \\ \neg f(Y) & \text{if } X \text{ has the form } \neg Y \\ f(Y \vee Z) & \text{if } X \text{ has the form } Y \vee Z \\ f(Y \wedge Z) & \text{if } X \text{ has the form } Y \wedge Z \\ f(Y \rightarrow Z) & \text{if } X \text{ has the form } Y \rightarrow Z \end{cases}$$

assuming a good input

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Mathematical Induction

- Theorem: **Every** wff in propositional logic has an equivalent form without ' \rightarrow ', using the equivalence between $X \rightarrow Y$ and $\neg X \vee Y$.
- Proof by **Mathematical Induction**
 - Base case (propositional variable): Done.
 - Induction step: Suppose that the theorem applies to all the components (ϕ and ψ) of the current formula.
 - Case 1 ($\neg\phi$): Done.
 - Case 2 ($\phi \wedge \psi$): Done.
 - Case 3 ($\phi \vee \psi$): Done.
 - Case 4 ($\phi \rightarrow \psi$): Equivalent to $\neg\phi \vee \psi$.

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Section Summary

- Inductive techniques are useful for defining and proving unbounded (potentially infinite) things in a finite manner.
- They can be used
 - Constructively: Set definition
 - Analytically: Mathematical induction
 - In both ways: Function definition and use

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Section 3

Deduction and Induction

- **Induction** [in general]: Argument in the direction of **specific** to **general** Always true?
 - E.g., By looking at a lot of birds in a park, conclude that "all birds can fly."
- **Deduction** [in general]: Argument in the direction of **general** to **specific**
 - E.g., From the fact that all living things will die, we conclude that "I will die."

More examples?
Examples in logic?

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Deduction in Logic

- A proof of a theorem (special) within a logical system (general) is an example of deduction.

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Induction in Logic

- Inductive/recursive definition: A definition of a potentially infinite set/function (general) based on a finite number of statements (special)
- **Mathematical Induction**: A proof technique that is used to justify an infinite number of cases (i.e., general) based on a finite number of statements (special)

Proof of Mathematical Induction?

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Section Summary

- Deduction: general to specific
 - Always true
- Induction: specific to general
 - Not always true
 - We accept **certain specific patterns**, e.g., mathematical induction.

Summary Exercise

- Explain the similarities and differences between the following:
 - Inductive definition of sets
 - Recursive definition of functions
 - Mathematical induction
- **Questions/Comments/Suggestions**