Announcements

- Final Evaluation Workshop
  - Section 1: 8:00-10:00 a.m. in HH253
  - Section 2: 11:00 a.m.-1:00 p.m. in HH253
  - 4 sessions for 4 exercises
  - No mutual evaluation of Mini Projects
- Final Comprehensive Exercises
  - To be available on-line on Thursday
  - Recommended to preview and ask questions on Friday

Related problem in FinComp

Producer-Consumer

- Condition for termination?

$$produce(m) = \begin{cases} m & \text{if } m > 9 \\ consume(m \times 1.25) & \text{otherwise} \end{cases}$$

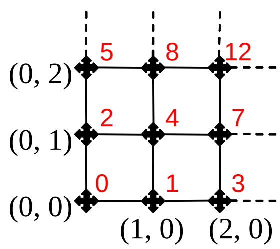
$$consume(m) = \begin{cases} m & \text{if } m < 1 \\ produce(m \times 0.75) & \text{otherwise} \end{cases}$$

Mutual recursion

How to prove it?

Review: Unit D2 Ex1

Labeling stations sequentially



Related problem in FinComp

Stations

C	0	1	2	3	4	5	6	7	8
0	√	√	√	√	√	√	√	√	√
1	√	√	√	√	√	√	√	√	√
2	√	√	√	√	√	√	√	√	√
3	√	√	√	√	√	√	√	√	√
4	√	√	√	√	√	√	√	√	√
5	√	√	√	√	√	√	√	√	√
6	√	√	√	√	√	√	√	√	√
7	√	√	√	√	√	√	√	√	√
8	√	√	√	√	√	√	√	√	√

Reflexive, symmetric, transitive? Visualize?

Reflexive

C	0	1	2	3	4	5	6	7	8
0	√								
1		√							
2			√						
3				√					
4					√				
5						√			
6							√		
7								√	
8									√

Symmetric

C	0	1	2	3	4	5	6	7	8
0		√							
1	√		√						
2		√							
3					√	√		√	
4				√			√		
5				√			√		
6					√	√			
7				√				√	
8									√

Transitive

C	0	1	2	3	4	5	6	7	8
0		√	√						
1			√						
2									
3				√					
4							√	√	
5									
6									
7							√		
8									

Equivalence Relation

C	0	1	2	3	4	5	6	7	8
0	√	√	√						
1	√	√	√						
2	√	√	√						
3				√					
4					√	√	√	√	
5					√	√	√	√	
6					√	√	√	√	
7					√	√	√	√	
8									√

Union of exhaustive sub-Cartesian products

Communication loss

C	0	1	2	3	4	5	6	7	8
0	√	√	√	√	√	√	√	√	√
1	√	√	√	√	√	√	√	√	√
2	√	√	√	√	√	√	√	√	√
3	√	√	√	√		√	√	√	√
4	√	√	√	√	√	√	√	√	√
5	√	√	√	√	√	√	√	√	√
6	√	√	√	√	√	√	√	√	√
7	√	√	√	√	√	√	√	√	√
8	√	√	√	√	√	√	√	√	√

Symmetric effect (one-way transmission is not communication)

C	0	1	2	3	4	5	6	7	8
0	√	√	√	√	√	√	√	√	√
1	√	√	√	√	√	√	√	√	√
2	√	√	√	√	√	√	√	√	√
3	√	√	√	√		√	√	√	√
4	√	√	√		√	√	√	√	√
5	√	√	√	√	√	√	√	√	√
6	√	√	√	√	√	√	√	√	√
7	√	√	√	√	√	√	√	√	√
8	√	√	√	√	√	√	√	√	√

Even indirect communication is blocked.

Impossible:
- (3, 0) + (0, 4)
- (3, 1) + (1, 4)
- (3, 2) + (2, 4)
- (3, 3) + (3, 4)
- (3, 4) + (4, 4)
- ...
- (3, 8) + (8, 4)
- (4, 0) + (0, 3)
- ...
- (4, 8) + (8, 3)

C	0	1	2	3	4	5	6	7	8
0	√	√	√	√	√	√	√	√	√
1	√	√	√	√	√	√	√	√	√
2	√	√	√	√	√	√	√	√	√
3	√	√	√	√		√	√	√	√
4	√	√	√		√	√	√	√	√
5	√	√	√	√	√	√	√	√	√
6	√	√	√	√	√	√	√	√	√
7	√	√	√	√	√	√	√	√	√
8	√	√	√	√	√	√	√	√	√

Remove all the two-step links

Impossible:
- (3, 0) + (0, 4)
- (3, 1) + (1, 4)
- (3, 2) + (2, 4)
- (3, 3) + (3, 4)
- (3, 4) + (4, 4)
- ...
- (3, 8) + (8, 4)
- (4, 0) + (0, 3)
- ...
- (4, 8) + (8, 3)

C	0	1	2	3	4	5	6	7	8
0	√	√	√			√	√	√	√
1	√	√	√			√	√	√	√
2	√	√	√			√	√	√	√
3				√					
4					√				
5	√	√	√			√	√	√	√
6	√	√	√			√	√	√	√
7	√	√	√			√	√	√	√
8	√	√	√			√	√	√	√

Rearranged: Still an equivalence relation

C	0	1	2	5	6	7	8	3	4
0	√	√	√	√	√	√	√		
1	√	√	√	√	√	√	√		
2	√	√	√	√	√	√	√		
5	√	√	√	√	√	√	√		
6	√	√	√	√	√	√	√		
7	√	√	√	√	√	√	√		
8	√	√	√	√	√	√	√		
3								√	
4									√

Alternative: Blocking just one of the two steps?

Impossible:
- (3, 0) + (0, 4)
- (3, 1) + (1, 4)
- (3, 2) + (2, 4)
- (3, 3) + (3, 4)
- (3, 4) + (4, 4)
- ...
- (3, 8) + (8, 4)
- (4, 0) + (0, 3)
- ...
- (4, 8) + (8, 3)

C	0	1	2	3	4	5	6	7	8
0	√	√	√	√	√	√	√	√	√
1	√	√	√	√	√	√	√	√	√
2	√	√	√	√	√	√	√	√	√
3	√	√	√	√		√	√	√	√
4	√	√	√		√	√	√	√	√
5	√	√	√	√	√	√	√	√	√
6	√	√	√	√	√	√	√	√	√
7	√	√	√	√	√	√	√	√	√
8	√	√	√	√	√	√	√	√	√

Remove one of the two steps: Still an equivalence relation

Impossible:
- (3, 0) + (0, 4)
- (3, 1) + (1, 4)
- (3, 2) + (2, 4)
- (3, 3) + (3, 4)
- (3, 4) + (4, 4)
- ...
- (3, 8) + (8, 4)
- (4, 0) + (0, 3)
- ...
- (4, 8) + (8, 3)

C	0	1	2	3	4	5	6	7	8
0	√	√	√	√					
1	√	√	√	√					
2	√	√	√	√					
3	√	√	√	√					
4					√	√	√	√	√
5					√	√	√	√	√
6					√	√	√	√	√
7					√	√	√	√	√
8					√	√	√	√	√

Unit D5: Logic

Focus on Mathematical Induction

Today

- Understand and use Mathematical Induction
- Compare inductive techniques
- Understand the general meaning of deduction and induction
- Take-home exercises
  - Highway, Teriyaki Boy, Program analysis

Section 1

Sum of Linear Series

$$\sum_{i=1}^{n} i = 1 + 2 + \ldots + n = \frac{n \times (n+1)}{2}$$

- Gauss' story
- Mathematical induction

Mathematical Induction

- A widely accepted proof technique
- Let **N** = {0, 1, 2, 3, ...}
- Let $P(n)$ be a unary relation (set) on $n \in$ **N**
- If both of the following conditions hold:
  - $P(n_0)$, where $n_0 \in$ **N** (e.g., $n_0 = 0$)
  - For every $n \geq n_0$ ($n \in$ **N**), $P(n)$ implies $P(n + 1)$
- Then, $P(n)$ is true for every $n \geq n_0$ ($n \in$ **N**)

Math Induction Proof Pattern

- Main hypothesis: None
- Main conclusion: $\Sigma_{1 \le i \le n}\, i = n \times (n + 1) / 2$
- Base case ($n = 1$): $\Sigma_{1 \le i \le 1}\, i = 1 = 1 \times (1 + 1) / 2$
- Induction step
  - Induction hypothesis: $\Sigma_{1 \le i \le n}\, i = n \times (n + 1) / 2$
  - Conclusion: $\Sigma_{1 \le i \le (n+1)}\, i = (n + 1) \times (n + 2) / 2$
  - Proof (of the induction step)
    1. $\Sigma_{1 \le i \le (n+1)}\, i = \Sigma_{1 \le i \le n}\, i + (n + 1)$ [LHS Conclusion]
    2. $\Sigma_{1 \le i \le n}\, i + (n + 1) = n \times (n + 1) / 2 + (n + 1)$ [Ind. hyp.]
    3. $n \times (n + 1) / 2 + (n + 1) = (n + 1) \times (n + 2) / 2$ [Arithmetic]
    4. $\Sigma_{1 \le i \le (n+1)}\, i = (n + 1) \times (n + 2) / 2$ [Transitivity eq/ineq: 1.-3.]
- By Math Induction (shaded), main conclusion holds.

Pigeonhole Principle

- Theorem: If $n + 1$ letters are put in n mailboxes, then some mailbox will contain (at least) two letters.
- Proof (for any natural number $n > 0$)
  - Base case ($n = 1$): The mailbox contains two letters.
  - Induction step
    - Induction hypothesis: The theorem holds for n.
    - Conclusion: It holds for $n + 1$ as well.
      - Proof (of the induction step)
      - Case 1 (first mailbox has 2 letters): Done.
      - Case 2 (otherwise): ______________
  - By Mathematical Induction, the theorem holds.

Group Exercise 1

- $(1 + 2 + ... + n)^2 = 1^3 + 2^3 + ... + n^3$ for any positive integer n

Cardinality of Power Sets

- Prove that $|P(S)| = 2^{|S|}$ for any set S (including $\varnothing$)
  - $P(S)$: the power set of a set S
  - $|S|$: the cardinality of set S
- Hint: Induction on the size of S

Section 2

Inductive Techniques

- Inductive definition of sets
- Recursive definition of functions
- Mathematical induction (a proof technique)
- Common feature
  - Base case
  - Induction step

Finite means to deal with unbounded situations (potential infinity)

Also “Exclusion clause” for set definition

Inductive Set Definition

Example

- wff’s in propositional logic
  - Base: Propositional variables are a wff.
  - Induction step:
    - If φ is a wff, $(\neg\varphi)$ is a wff.
    - If φ and ψ are wff’s, $(\varphi \wedge \psi)$, $(\varphi \vee \psi)$, and $(\varphi \rightarrow \psi)$ are a wff.
  - Exclusion: Nothing else is a wff.

Exclude ‘$\leftrightarrow$’

Recursive Function Definition

- Replace all the patterns of $X \to Y$ to $\neg X \vee Y$

$$f(X) = \begin{cases} X & \text{if } X \text{ is a propositional variable} \\ \neg f(Y) & \text{if } X \text{ has the form } \neg Y \\ & \text{if } X \text{ has the form } Y \vee Z \\ & \text{if } X \text{ has the form } Y \wedge Z \\ & \text{if } X \text{ has the form } Y \to Z \end{cases}$$

assuming a good input

Mathematical Induction

- Theorem: Every wff in propositional logic has an equivalent form without '$\to$', using the equivalence between $X \to Y$ and $\neg X \vee Y$.
- Proof by Mathematical Induction
  - Base case (propositional variable): Done.
  - Induction step: Suppose that the theorem applies to all the components (φ and ψ) of the current formula.
    - Case 1 ($\neg\varphi$): Done.
    - Case 2 ($\varphi \wedge \psi$): Done.
    - Case 3 ($\varphi \vee \psi$): Done.
    - Case 4 ($\varphi \to \psi$): Equivalent to $\neg\varphi \vee \psi$.

Section Summary

- Inductive techniques are useful for defining and proving unbounded (potentially infinite) things in a finite manner.
- They can be used
  - Constructively: Set definition
  - Analytically: Mathematical induction
  - In both ways: Function definition and use

Section 3

Deduction and Induction

- Induction [in general]: Argument in the direction of specific to general

  Always true?
  - E.g., By looking at a lot of birds in a park, conclude that "all birds can fly."
- Deduction [in general]: Argument in the direction of general to specific
  - E.g., From the fact that all living things will die, we conclude that "I will die."

More examples?
Examples in logic?

Deduction in Logic

- A proof of a theorem (special) within a logical system (general) is an example of deduction.

Induction in Logic

- Inductive/recursive definition: A definition of a potentially infinite set/function (general) based on a finite number of statements (special)
- Mathematical Induction: A proof technique that is used to justify an infinite number of cases (i.e., general) based on a finite number of statements (special)

Proof of Mathematical Induction?

Section Summary

- Deduction: general to specific
  - Always true
- Induction: specific to general
  - Not always true
  - We accept certain specific patterns, e.g., mathematical induction.

Summary Exercise

- Explain the similarities and differences between the following:
  - Inductive definition of sets
  - Recursive definition of functions
  - Mathematical induction
- Questions/Comments/Suggestions