Ex B1

- Part 1 UTM Review question: describe UTM
  - Task 1: Circularity examples
  - Task 2: A: Compiling a TM, B: ID'ing TMs
  - Task 3: UTM in the reading
  - Task 4: TM version of the liar paradox; universality
- Part 2 Review

Ex B0 Part 1 Task 1 phrasing

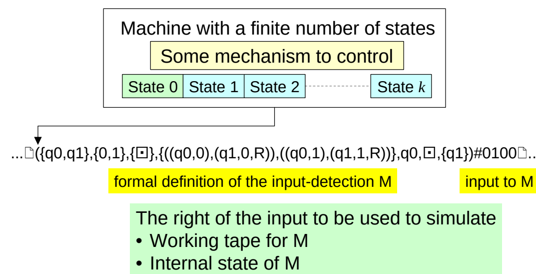
Universal TM (UTM)

- Given (i) an encoding of the formal definition of a TM, M, and (ii) an input string, i, to M on its tape, simulates the behavior of M on i.
  - I.e.. specification of a TM through the formal definition on the tape.
- Accepts if M accepts i.
  - I.e., exactly mirrors the behavior of M.

UTM Specifics

- Tape input
  - ...□*TM formal def*#*input to the TM*□...
- Mechanism
  - Allocate an unused area of the tape to represent the working tape for the TM (if more space is required, move the surrounding areas outward)
  - Allocate an unused area of the tape to keep record of the internal state of the TM
  - Interpret the TM definition and simulate its behavior on the input to the TM

UTM, Schematically



Unit B2: Overview

- Understand mathematical tools for understanding undecidability
  - Explore ID'ing TMs with natural numbers
  - Explore the notion of "counting"
  - Introduce the "diagonalization" technique
- Understand the "diagonalization language"
- Preview Exercise B2 "Diagonalization"

Labeling TMs

- Cf. Ex B1 Part 1 Task 2B
- Possible to assign a unique ID (natural number) to *every* TM?

Thinking Big ...

- How many TMs are there in the world?
- How many inputs must a TM deal with?
  - How many different possibilities must a TM deal with?
- Mismatch between the collection of TMs and the collection of input possibilities ⇔ Undecidability?
- Mathematical support for analyzing computability

Set Cardinality

Informally, the number of elements in a set

- Countable
  - Finite: e.g., $|\{yes, no\}| = 2$, $|\varnothing| = 0$
  - Countably infinite: e.g., **N** (natural numbers)
- Uncountable: If not countable

Group Exercise 1

- Analyze whether the following sets are countable
  1. Set of integers
  2. Set of rational numbers
  3. Set of real numbers
  4. Set of all TMs
  5. Set of all strings
  6. Set of all *languages* (set of sets of strings)
  7. *Power set* of some given set

Countable Sets

Possible to order along with natural numbers

- Any finite set
- Set of natural numbers
- Set of integers
- Set of rational numbers (represented as fractions)
- Set of all strings
- Set of all TMs

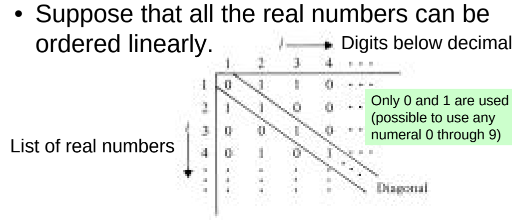
Uncountable Sets

Impossible to order along with natural numbers

- Set of real numbers
  - Diagonalization [Cantor]
- *Power set* of natural numbers
  - Same "density" as real numbers
- Set of all *languages* (i.e., set of sets of strings, or *power set* of all strings

Cantor's Diagonalization

- Suppose that all the real numbers can be ordered linearly.



- There still is a number not covered.
- The assumption is wrong. Impossible.

Proof by Contradiction

To prove: Statement X

- Assume the negation of X, i.e., *not* X
- Derive a contradiction
- Then, by this proof technique (proof by contradiction) X must be true (because X is either true or false)

Proof by Contradiction, Formally

Hypothesis: ***m*** « ***o***, Ø***m*** ® ***i***, Ø***i***

Proof of ***m*** from the above hypothesis

1.	Ø***m*** ® ***i***	[hyp]
2.	| Ø***m***	[hyp to be rejected]
3.	| ***i***	[Modus Ponens: 1, 2]
4.	| Ø***i***	[hyp]
5.	| **F**	[contradiction: 3, 5]
6.	***m***	[proof by contradiction: 2-5]

Power Set (of set A): $P(A)$

- The set of all the subsets of A, i.e., $P(A) = \{X \mid X \subseteq A\}$
- Examples
  - $P(\{0, 1\}) = \{\emptyset, \{0\}, \{1\}, \{0, 1\}\}$
  - $P(\{0\}) = \{\emptyset, \{0\}\}$
  - $P(\emptyset) = \{\emptyset\}$
- Cardinality: $|P(A)| = 2^{|A|}$
  - $|P(\{0, 1\})| = |\{0, 1\}|^2 = 2^2 = 4$
  - $|P(\mathbf{N})| = ?$

$P(A)$ always contains $\emptyset$ and A

Happens to be: $\emptyset = A$

Power set in the real world?

Interim Summary

- Power set always increases the cardinality.
- The power set of a countable set: uncountable
- The set of TMs: countable
- The set of strings: countable
- The set of languages (specifies which string to be included): power set of the set of strings ⇒ uncountable
- There won't be enough TMs to cover all languages ⇒ The source of undecidability

schematics

solution?

cf. pigeon hole principle

The halting problem explained?

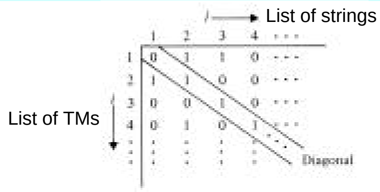
Agenda

- Rest of today: Identify an example language which *no* TM can decide (there must be uncountably many of those...)
- Next class: Using this language, prove the computability classes of the main problem (e.g., halting problem).

Group Exercise 2

- Identify an example language which *no* TM can decide (there must be uncountably many...)
- Hint: Consider the collection of all languages; Recall Cantor's diagonalization

Diagonalization Language (L_d)



- Entry 1: M_i accepts w_j
- Entry 0: M_i does not accept w_j
- $L_d = \{w_i \mid M_i$ does not accept $w_i\}$ No TM accepts!

Diagonalization Language

- The diagonalization language is non-TM-recognizable because no TM would accept it.

Main Problems

- The halting problem is semi-decidable.
  $HALT_{TM} = \{(M, w) \mid$ TM M halts on $w\}$ *w* for "word" (input)
- Infinite loop detection is unsolvable.
  $LOOP_{TM} = \{(M, w) \mid$ TM M loops on $w\} = (HALT_{TM})'$
- Universal language is semi-decidable.
  $ACCEPT_{TM} = \{(M, w) \mid$ TM M accepts $w\}$
- The complement of universal language is unsolvable.
  $NACCEPT_{TM} = \{(M, w) \mid$ TM M does *not* accept $w\} = (ACCEPT_{TM})'$

A': set complement of *A*

Unit Summary

- Set cardinality: countable vs. uncountable
- Power set: increases the cardinality
- Diagonalization: shows uncountability of a set (using proof by contraction)
- The source of undecidability: uncountability
- The diagonalization language, L_d

Midterm Survey

- To be administered and collected by a volunteer