Ex B3: Mini Presentations

The main problems

- The universal language is semi-decidable.
  $ACCEPT_{TM} = \{(M, w) \mid \text{TM } M \text{ accepts } w\}$
- The complement of universal language is non-TM-recognizable.
  $NACCEPT_{TM} = \{(M, w) \mid \text{TM } M \text{ does } not \text{ accept } w\}$
- The halting problem is semi-decidable.
  $HALT_{TM} = \{(M, w) \mid \text{TM } M \text{ halts on } w\}$
- Infinite loop detection is non-TM-recognizable.
  $LOOP_{TM} = \{(M, w) \mid \text{TM } M \text{ loops on } w\}$

Connection between $ACCEPT_{TM}$ and $HALT_{TM}$?

$ACCEPT_{TM}$ vs. $HALT_{TM}$

- Both are properties of the set of TMs and undecidable
  - Property: Division of a set, i.e., a subset
  - E.g., “concrete” (set of concrete things among everything, cf. abstract things)
- Each has a different decision criterion

Would other decision criteria on TMs also lead to undecidability?

Properties of TMs/Languages

- What a TM can recognize = Language
  - I.e., fix a TM ⇒ The language is fixed.
- Property of TMs/languages
  - A subset of all the TMs/languages
  - The power set of TMs/languages is uncountable.
  - Division of an uncountable set ⇒ Will always leave out an uncountable part (undecidable)

Example Properties

- Universal = $ACCEPT_{TM}$
- Terminating = $HALT_{TM}$
- Finite = $\{L \mid L \text{ is finite}\}$
  ~ $\{M \mid \text{TM } M \text{ accepts a finite language}\}$
- Regular = $\{L \mid L \text{ is regular}\}$
  ~ $\{M \mid \text{TM } M \text{ accepts a regular set}\}$
- Property X = $\{L \mid L \text{ has Property X}\}$
  ~ $\{M \mid \text{TM } M \text{ recognizes Property X}\}$

Rice’s Theorem

- Non-trivial properties of languages/TMs are undecidable ⇐ Uncountability
  - Note: Trivial properties (e.g., emptiness) that do not divide the entire set is trivially decidable.
- Potentially useful for Ex B5 (Comprehensive Ex)

Text Section 9.3.3

Dealing with Multiple Problems

- Consider: $ACCEPT_{TM}$ and $HALT_{TM}$
- Benefits of re-using known problems, e.g., in solving new problems

Unit B4: Overview

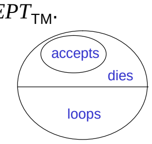
- Discuss a way to use one problem to analyze another problem
  - Understand problem transformation
  - Understand how certain properties of problems can be transferred
  - Discuss equivalence of problems
- Preview Exercise B4 "Reduction/Equivalence"

Review

Subproof Idea

To show: $HALT_{TM}$ is undecidable.

- Proof by contradiction
  - Suppose that $HALT_{TM}$ is *decidable*.
  - If "halt" is detected, accept/reject based on the UTM ($ACCEPT_{TM}$); otherwise, reject.
  - This is an algorithm for $ACCEPT_{TM}$.
  - A contradiction



Underlying Idea

- Suppose that $HALT_{TM}$ is decidable.
- To show that $ACCEPT_{TM}$ is also decidable, using the hypothesis.
  - I.e., simulate $ACCEPT_{TM}$ using the hypothetical, deciding TM for $HALT_{TM}$.

Schematic

Transferring Decidability?

- Consider problems **A** and **B**
- Would the following be correct?

A is *transformed* to **B** and **B** is decidable.
⇒ **A** is decidable as well.

Transferring Undecidability?

- Consider problems **A** and **B**
- Would the following be correct?

A is undecidable and is *transformed* to **B**.
⇒ **B** is undecidable as well.

Precise notion of such transformation?

Problem Transformation

- Motivations
  - Connect different problems, one of which has known properties
  - Identify applicable properties in other problems
- Requirement
  - Should maintain important properties

Reduction

- **A** *is reducible to* **B**.
  - ⇔ **A** is no more difficult than **B**.
  - ⇔ If one can solve **B**, one can solve **A**.
  - ⇔ If one can't solve **A**, one can't solve **B**.
  - ⇔ **A** can be simulated by **B**.
- Idea: One *cannot* solve a difficult problem with an easy means.

Different types of reducibility: e.g., mapping vs. Turing reducibility

Practice: Correct Reduction?

1. The state's daily deficit of 11 million units can be reduced to just three million.
2. A paper is reducible to one sentence.
3. Biology is reducible to physics.
4. A schematic diagram can be reduced to a set of logical propositions.
5. Goodness is reducible to betterness.

Theorems

- **A** is reducible to **B** and **B** is decidable.
  - ⇒ **A** is decidable as well.
    - Justification: By definition (decidability of **A** depends on that of **B**)
- **A** is reducible to **B** and **A** is undecidable.
  - ⇒ **B** is undecidable as well.
    - Justification: Proof by contradiction

Reduction Examples

Incorrect

- If $ACCEPT_{TM}$ is reducible to $HALT_{TM}$ and $HALT_{TM}$ is decidable, $ACCEPT_{TM}$ is also decidable.
- If $HALT_{TM}$ is reducible $ACCEPT_{TM}$ to and $HALT_{TM}$ is undecidable, $ACCEPT_{TM}$ is also undecidable.

Correct: reduction with respect to computability analysis

- If $ACCEPT_{comp}$ is reducible to $HALT_{comp}$ and $HALT_{TM}$ is decidable, $ACCEPT_{TM}$ is also decidable.
- If $HALT_{comp}$ is reducible $ACCEPT_{comp}$ to and $HALT_{TM}$ is undecidable, $ACCEPT_{TM}$ is also undecidable.

- If your problem is reducible to my problem and my problem is acceptable, your problem is also acceptable.

Theorems (generalized)

- **A** is reducible to **B** and **B** has a positive property (e.g., decidable, TM-recognizable).
  - ⇒ **A** has the positive property as well.
- **A** is reducible to **B** and **A** has a negative property (e.g., undecidable, non-TM-recognizable).
  - ⇒ **B** has the negative property as well.
- Positive:
- Negative:

Problem Equivalence

- Two problems are equivalent if they are reducible to each other.
- Two mechanisms are equivalent if they can solve exactly the same set of problems.
  - Equivalently, if they can simulate each other.

Equivalence Examples

Incorrect

- $ACCEPT_{TM}$ and $NACCEPT_{TM}$ are equivalent.
- $ACCEPT_{TM}$ and $HALT_{TM}$ are equivalent.

Correct

- $ACCEPT_{comp}$ and $NACCEPT_{comp}$ are equivalent.
- $ACCEPT_{comp}$ and $HALT_{comp}$ are equivalent.

- TMs with a semi-infinite tape are equivalent to standard TMs.
- Correct? TMs are equivalent to computers. [to be discussed]
- A computational decision problem is equivalent to a set membership problem.

Some qualifications

Unit Summary

- Understand the notion of reduction as problem transformation
  - Need at least as a powerful problem/mechanism to solve the original
  - Positive/negative properties can be analyzed.
- Understand the notion of problem equivalence
  - Through set comparison
  - Through machine comparison (simulation)

Summary Question

- For the past few class meetings, we have been discussing technical topics. It is natural that you have questions and feel uncertainty. What are they?