Unit D3 Supplement, 4/13/05

QBF Problem

The quantified variables in a QBF statement, i.e., *x*, *y*, *z*, ..., can refer to any propositional variables. But since a propositional variable can take either true or false, we can analyze the truth-value assignment of a QBF statement with respect to possible values of true or false. Let us consider the example introduced in class:

For all *x*, some *y* ((*x* or (not *y*)) and ((not *x*) or *y*))
[formally] $\forall x \, \exists y \, ((x \vee \neg y) \wedge (\neg x \vee y))$

To check $\forall x$, we will need to consider both x = true and x = false. Let us first set x = true. Then, if we choose y = true, the entire statement can be made true. Next, let x = false. Then, we can choose y = false to make the entire statement true. Thus, this statement holds. The strategy of choosing values is analogous to the game-theoretic interpretation of FOL (although in FOL, we choose an individual or value in a more general sense, not true/false).

When all the quantifiers are existential and the rest of the statement is organized as a conjunction of (disjunctive) clauses, the problem degenerates to SAT. So, if we can solve QBF, we can solve SAT. Thus, SAT reduces to QBF.

QBF is NP-hard (i.e., it can solve any NP problem). SAT can be reduced to QBF. Since SAT is in NPC, QBF must at least as difficult as SAT. So, any problem that SAT can solve must be solved by QBF. Note that the (positive) property that SAT is in NP does not transfer to QBF.

However, in the worst case, all the quantifiers can be universal. Then, we need to check all the combinations of truth values. So, even to verify an answer, it will take exponential time, as shown in an example with 3 variables below.

Combo	x_1	x_2	x_3
1	T	T	T
2	T	T	F
3	T	F	T
4	T	F	F
5	F	T	T
6	F	T	F
7	F	F	T
8	F	F	F

QBF is not in NP. Due to the quantifiers (the universal quantifier in particular), it is not possible to verify the truth value of a given statement in polynomial time. That is, even to verify a correct answer, we will still need to check all the combinations of truth-value assignment. Thus, it is worse than NP.

QBF is in PSPACE. Given any QBF statement, we can analyze its truth value in a way analogous to the analysis of the example statement above. Suppose that there are three universally quantified variables, x_1, x_2, and x_3. First, set x_1 = true in the working tape position 1. Then, set x_2 = true in the working tape position 2, and so on, as shown schematically below.

x_1	x_2	x_3

If this assignment result in false, the entire statement cannot be true (because all the combinations must result in true). Using a recursion, we can cover all the combinations. So, we need $O(n)$ working tape space, which is Polynomial.

Time-Space Tradeoff

One natural idea about time-space tradeoff may be that with more space, one can reduce the time. Scott (in his summary question) represented this (very roughly) as: Cost = Time $\times$ Space. Consider the following example. You open a computer repair shop in your own dorm room. If ten customers bring in ten computers, your work efficiency may well suffer from the lack of space. On the other hand, if you had a typical classroom, you could do various things much more efficiently. The time-space tradeoff for computers is often discussed along this line.

Departing from this particular task, let us compare your dorm room and a typical classroom with respect to a broader range of tasks. If you need to vacuum, you will need more time for the classroom. If you need to keep the space warm, it will cost more to warm up the entire classroom. The discussion of the relation P $\subseteq$ PSPACE appears to be more related to this view. That is, with more space, you could potentially do more things, which would potentially take more time.

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